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Effet-Pinch Relativiste.

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Résumé. — En considérant la métrique relativiste de symétrie cylindrique (15) qui généralise, par example, la métrique du disque tournant ([10-12]), on peut étudier avec nos méthodes antérieures ([1-6]) les extrémales et la dynamique attachée à cette métrique. En soulignant que W/m_0 est l'expression de l'énergie potentielle de l'unité de masse, c'est-à-dire $W = -m_0 V$, m_0 étant la masse de repos du corpuscule se mouvant dans le champ représenté, au point de vue classique, par le potentiel $V(r)$, les calculs qu'on suit assez bien dans le texte, nous donnent l'expression approximative (25) de la force relativiste électrodynamique, force de nature pondéro-motrice, agissant perpendiculairement à l'axe du cylindre sur une charge du plasma se trouvant à l'intérieur de l'installation cylindrique. La vitesse de la charge étant donnée par (23) on suppose que $\dot{r}=v$, $|W| \ll m_0 c^2$ et $|v| < c$. En dehors du terme classique en W', l'expression de la force contient alors un terme en W/r qui représente une force corrective attractive ou répulsive suivant le signe de la charge qui peut contribuer à la contraction du courant, à la production du phénomène de striction (pinch-effect). On signale, en même temps, aussi le cas des installations toroïdales.

Pour étudier, au point de vue relativiste, la dynamique d'un point matériel, en mouvement dans un champ, dérivant d'un potentiel scalaire, il faut déterminer la métrique, c'est-à-dire le ds^2 relativiste. Dans certains cas on arrive à la solution du problème, en résolvant les équations d'Einstein. Mais cette méthode présente des difficultés dans le cas des champs électromagné-

tiques, mésoniques etc., puisque on ne peut pas donner la généralisation correspondante, acceptable pour tout le monde, des équations d'Einstein. Il existe, pourtant ([1-3]), une possibilité d'obtenir la métrique, directement, à l'aide des considérations géométriques ([*]), en se basant sur le *principe de l'équivalence* des différents champs (gravifiques, d'inertie, électriques, mésoniques, etc.), en ce qui concerne la description des phénomènes physiques. En effet, n'importe quelle métrique relativiste doit posséder les propriétés suivantes:

Le $\mathrm{d}s^2$ relativiste

$$\mathrm{d}s^2 = g_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k \,. \tag{1}$$

a) tend asymptotiquement vers le $\mathrm{d}s^2$ de la métrique pseudo-euclidienne de Poincaré-Minkowski, dans les points situés à grande distance de la matière active (masses ou charges, qui donnent naissance au champ mentionné);

b) au voisinage de la matière active, les phénomènes physiques, décrits par (1), doivent être étudiés, pour être mesurables, dans la variété pseudo-euclidienne localement tangente à (1);

c) les composantes du tenseur métrique fondamental g_{ik} doivent satisfaire au principe d'équivalence, mentionné antérieurement.

En ce qui concerne le $\mathrm{d}s^2$ de la variété pseudo-euclidienne de Poincaré-Minkowski

$$\mathrm{d}s^2 = c^2\,\mathrm{d}T^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \equiv c^2\,\mathrm{d}T^2 - \mathrm{d}R^2 - R^2\,\mathrm{d}\Theta^2 - R^2 \sin\Theta\,\mathrm{d}\Phi^2 \,, \tag{2}$$

on sait ([4]), que: (1) est un invariant du groupe de transformation d'Einstein-Lorentz; et (2) détermine la mesure du temps et de l'espace

$$\mathrm{d}t = \frac{\mathrm{d}T}{\sqrt{1 - v^2/c^2}}\,, \qquad \mathrm{d}l = \sqrt{1 - \frac{v^2}{c^2}}\,\mathrm{d}L \,, \tag{3}$$

caracterisée par la dilatation des durées (ralentissement des horloges) et la contraction des longueurs dans le sens du mouvement.

([1]) T. T. VESCAN: *Acta Bolyaiana, Cluj.*, **2**, 12 (1948).
([2]) T. T. VESCAN, A. WEISZMANN et J. GOTTLIEB: *Acad. RPR, Timişoara, Lucr. conf. geom. dif.*, 341 (1955).
([3]) T. T. VESCAN: *Anal. scient. Univ. Iassy*, **6**, 331 (1960).
([*]) Géométriques au point de vue purement formel, mais en essence de nature physique.
([4]) T. T. VESCAN: *Anal. scient. Univ. Jassy*, **6**, 101 (1960).

En plus, on a les conclusions suivantes:

α) les géodésiques de longueur nulle: $ds^2 = 0$ donnent la propagation de la lumière;

β) les géodésiques pour lesquelles: $ds^2 > 0$ donnent les lois du mouvement d'un point isolé.

On peut préciser la condition *b*) par les suivantes:

b') Pour obtenir d'un ds^2 quelconque le ds^2 de la variété pseudo-euclidienne tangente, on fixe les valeurs de x^i, on considère comme variables les différentielles dx^i et on obtient ainsi un ds^2 à coefficients constants, réductible à (2), qui définit ainsi, exactement, au voisinage de la matière active, le comportement physique des points matériels isolés.

Le condition *c*) demande une explication supplémentaire, qui est la suivante:

c') En comparant (2) et (3) tout se passe comme si, *localement*, la métrique était définie par

$$ds^2 = c^2 \left(1 - \frac{v^2}{c^2}\right) dt^2 - \frac{dr^2}{1 - v^2/c^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 \tag{4}$$

dans le cas de la symétrie radiale, et par

$$ds^2 = c^2 \left(1 - \frac{v^2}{c^2}\right) dt^2 - \frac{r^2 d\theta^2}{1 - v^2/c^2} - dr^2 - dz^2 \tag{5}$$

dans le cas de la symétrie cylindrique etc. Nous reviendrons, d'ailleurs, sur cette question. Maintenant, en observant que

$$J_c = \tfrac{1}{2} m_0 v^2 \tag{6}$$

est l'expression de l'énergie cinétique classique, ou bien

$$J_r = m_0 c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right], \tag{7}$$

l'expression de l'énergie cinétique relativiste d'un point matériel, on peut exprimer par

$$J + W = E = \text{const.} \tag{8}$$

le principe de la conservation de l'énergie et obtenir ainsi, une voie géométrique convenable pour la détermination de la métrique relativiste, sans utiliser di-

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rectement les équations d'Einstein ou ses généralisations plus ou moins arbitraires. On obtient ainsi, par exemple ([5]), le ds^2 de Schwarzschild, en écrivant que dans un champ gravitationnel, de symétrie centrale, on a

$$\frac{1}{2} m_0 v^2 = k \frac{m_0 M}{r} + \text{const} \,,$$

et on observant que pour $r \to \infty$, on a $v \to 0$, c'est-à-dire que la constante est nulle. On arrive donc à la conclusion que

$$\frac{v^2}{c^2} = \frac{2kM}{rc^2} \,, \tag{9}$$

et en confrontant (4) et (9), précisement, à

$$ds^2 = c^2 \left(1 - \frac{2kM}{rc^2}\right) dt^2 - \frac{dr^2}{1 - 2kM/rc^2} - r^2 \, d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 \,, \tag{10}$$

l'expression du ds^2 extérieur de Schwarzschild. Cette méthode de géometrisation permet d'écrire d'une manière plus générale que

$$\tfrac{1}{2} m_0 v^2 = - W(r) = m_0 V(r)$$

et donc on aura

$$ds^2 = c^2 \left(1 - \frac{2V}{c^2}\right) dt^2 - \frac{dr}{1 - 2V/c^2} - r^2 \, d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 \,. \tag{11}$$

Nous supposons que cette métrique généralisée, (11), est valable pour n'importe quel champ de symétrie radiale, quelque soit la nature de la masse active (masse gravitationnelle, charge électrique etc.). C'est une hypothèse physique, qui se vérifie par ses conséquences ([6]). Nos collaborateurs, Z. Gábos ([7]) et J. Gottlieb ([8]), ont reussit d'obtenir, d'autres métriques, avec des conséquences remarquables au point de vue de leurs applications physiques.

Pour ne pas allonger notre exposé, nous nous bornons ici à présenter la possibilité d'étudier le phénomène de « pincement » ou striction—(le « pinch-

([5]) T. T. Vescan, E. Mihul et G. Ioniţă: *Acad. RPR, Jassy, St. si cerc. st. S. mat. fiz. chim. st. tehn.*, 6, 217 (1955).

([6]) T. T. Vescan: *Compt. Rend.*, **245**, 2014 (1957); **247**, 2301 (1958).

([7]) Z. Gábos: *Acad. RPR, Jassy, St. si cerc.*, 6, 110 (1953).

([8]) J. Gottlieb: *Nuovo Cimento*, **14**, 1166 (1959).

effect ») — présenté par les générateurs thermo-nucléaires, c'est-à-dire, exactement, de relever les aspects relativistes du phénomène de contraction du courant de décharge à haute-énergie.

D'après EINSTEIN [9], la fréquence d'un signal provenant du centre d'un disque tournant, ayant la vitesse angulaire ω, à la distance r du centre est

$$\nu = \nu_0 \sqrt{1 - \frac{v^2}{c^2}} = \nu_0 \sqrt{1 - \frac{\omega^2 r^2}{c^2}} = \nu_0 \sqrt{1 + \frac{2V}{c^2}}\,, \tag{12}$$

où

$$V = -\tfrac{1}{2}\omega^2 r^2 \tag{13}$$

et l'expression de la force centrifuge

$$f = -m_0 \frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\mathrm{d}W}{\mathrm{d}r}\,, \qquad \text{avec } W = -m_0 V, \tag{14}$$

W étant l'énergie potentielle de la masse ponctuelle m_0, située à la distance r du centre. En comparant (3), (5), (12) et (14) on peut dire que l'expression de la métrique relativiste généralisée de symétrie cylindrique sera donnée par

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2W}{m_0 c^2}\right)\mathrm{d}t^2 - \mathrm{d}r^2 - \frac{r^2\mathrm{d}\theta^2}{1 - 2W/m_0c^2} - \mathrm{d}z^2\,, \tag{15}$$

résultat énoncé, à l'aide de considérations différentes, par plusieurs auteurs [10-12] pour le cas particulier du disque tournant. Nous supposons, ici, que $W(r)$ est une fonction *quelconque* de r, hypothèse totalement nouvelle.

En appliquant le principe variationnel

$$\delta\int \mathrm{d}s = 0\,, \tag{16}$$

on obtient, aisément, les équations des extrémales

$$\frac{\partial\psi}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}\psi}{\mathrm{d}q_i'}\right] = 0\,, \qquad \left| \begin{array}{l} i = 1, 2, 3, 4;\ \text{avec:} \\ q_1 = r,\ q_2 = \theta,\ q_3 = z,\ q_4 = t, \end{array}\right. \tag{17}$$

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[9] A. EINSTEIN: *Über die spezielle und allgemeine Relativitätstheorie* (Berlin, 1920), p. 89.

[10] M. V. LAUE: *La théorie de la relativité*, vol. 2 (Paris, 1926), p. 173.

[11] J. CHAZY: *La théorie de la relativité et la mécanique céleste*, vol. 2 (Paris, 1930), p. 173.

[12] M. A. TONNELAT: *Les principes de la théorie électromagnétique et de la relativité* (Paris, 1959), pp. 269-277.

où

$$(18)\qquad \begin{cases} \psi = c^2\left(1-\dfrac{2W}{m_0c^2}\right)t'^2 - r'^2 - \dfrac{r^2\theta'^2}{1-2W/m_0c^2} - z'^2\,; \\ \text{et} \\ r'=\dfrac{\mathrm{d}r}{\mathrm{d}s}\,, \qquad \theta'=\dfrac{\mathrm{d}\theta}{\mathrm{d}s}\,, \qquad z'=\dfrac{\mathrm{d}z}{\mathrm{d}s}\,, \qquad t'=\dfrac{\mathrm{d}t}{\mathrm{d}s}\,, \end{cases}$$

c'est-à-dire

$$(19)\qquad c^2\left(1-\frac{2W}{m_0c^2}\right)t' = A = \text{const}\,,$$

$$(20)\qquad \frac{r^2\theta'}{1-2W/m_0c^2} = C = \text{const}\,,$$

$$(21)\qquad z' = B = \text{const.}$$

et

$$(22)\qquad r'' = \frac{W'}{m_0}t'^2 + \frac{r\theta'^2}{1-2W/m_0c^2} + \frac{W'r^2\theta'^2}{m_0c^2(1-2W/m_0c^2)^2}\,.$$

En observant, que la vitesse u est donnée par

$$(23)\qquad u^2 = \dot r^2 + r^2\dot\theta^2 + \dot z^2$$

l'élimination des constantes A, B, C et de la variable auxiliaire s nous conduit aux expressions suivantes de la force généralisée

$$(24)\qquad \begin{cases} f_r = m_0(\ddot r - r\dot\theta^2) = \dfrac{2(u^2-\dot z^2-\dot r^2)W}{rc^2(1-2W/m_0c^2)} + \\ \qquad\qquad + W'\left\{1-\dfrac{1}{1-2W/m_0c^2}\left[\dfrac{2\dot r^2}{c^2}+\dfrac{\dot r^2+\dot z^2-u^2}{c^2(1-2W/m_0c^2)}\right]\right\}, \\ f_\theta = \dfrac{1}{r}m_0\dfrac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\theta) = -\dfrac{4W'r\dot r\dot\theta}{c^2(1-2W/m_0c^2)}\,; \\ f_z = m_0\ddot z + \dfrac{2W'\dot r\dot z}{c^2(1-2W/m_0c^2)}\,. \end{cases}$$

Conclusion.

En soulignant que W/m_0 est l'expression de l'énergie potentielle de l'unité de masse, c'est-à-dire $W = -m_0V(r)$, m_0 étant la masse du corpuscule se mouvant dans le champ représenté, au point de vue classique, par le potentiel

$V(r)$, ces calculs nous donnent donc l'expression approximative suivante:

$$f_r \simeq W'\left(1-\frac{3\dot{r}^2}{c^2}\right)-\frac{2\dot{r}^2}{c^2}\left(\frac{W}{r}\right), \tag{25}$$

de la force relativiste électrodynamique, force de nature pondéro-motrice, agissant perpendiculairement à l'axe du cylindre sur une charge du plasma se trouvant à l'intérieur de l'installation cylindrique (OGRA). La vitesse de la charge étant donnée par (25) on suppose que

$$u^2 \simeq \dot{z}^2\,, \qquad \dot{r}\neq 0\,; \qquad |W|\ll m_0c^2 \qquad \text{et} \qquad |\dot{r}|<c\,.$$

En dehors du terme classique en W', l'expression de la force contient un terme en W/r qui représente une force corrective attractive ou répulsive suivant le signe de la charge, qui peut contribuer à la contraction du courant, à la production du phénomène de striction (pinch-effect). En relevant les recherches de STEPA [13] on conclue que le mouvement relativiste d'une charge dans un champ électromagnétique de symétrie cylindrique est accompagné toujours de l'effet « pinch », établissant ainsi une explication plausible de la stabilisation du phénomène de striction (concentration du courant filiforme) dans les installations expérimentales (réacteurs thermonucléaires).

Notre collaborateur, M.me GEORGETTE IONIȚĂ, candidat aux sciences mathématiques et physiques, en partant de l'expression du ds^2 de symétrie toroïdale

$$ds^2 = c^2\left(1-\frac{2W}{m_0c^2}\right)dt^2-\frac{dr^2}{1-2Wm_0c^2}-r^2\,d\varphi^2-(R+r\cos\varphi)^2\,d\theta^2\,, \tag{26}$$

a obtenu pour l'expression de la force généralisée dirigée vers le cercle moyen d'un tore des rayons R et r, la valeur suivante:

$$f_r = W'\left(1-\frac{2W}{m_0c^2}\right)-\frac{3W'\dot{r}^2}{c^2(1-2W/m_0c^2)}-\frac{2Wr\dot{\varphi}^2}{c^2}+\frac{2W\cos\varphi(R+r\cos\varphi)}{c^2}\,\dot{\theta}^2\,. \tag{27}$$

Cette expression est utilisable de la même manière que notre formule (24) et mérite l'attention des expérimentateurs.

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(13) N. I. STEPA: Recension de son travail par *Journ. de phys. techn. de l'URSS*, vol. **29**, n. 11, 1346 (1959); donnée en *Revista de referate. Fizica*, Inst. de studii rom. sov. Buc., **3**, 21 (1960).

RIASSUNTO (*)

Considerando la metrica relativistica a simmetria cilindrica (15) che generalizza, per esempio, la metrica del disco rotante [10-12], si possono studiare con i nostri precedenti metodi [1-6] gli estremali e la dinamica relativa a questa metrica. Sottolineando che W/m_0 è l'espressione dell'energia potenziale dell'unità di massa, cioè $W = -m_0 V$, in cui m_0 è la massa in quiete del corpuscolo che si muove nel campo rappresentato, dal punto di vista classico, dal potenziale $V(r)$, i calcoli, che si possono seguire molto bene nel testo, ci danno l'espressione approssimativa (25) della forza elettrodinamica relarivistica, forza di natura ponderomotrice, che agisce perpendicolarmente all'asse del cilindro su una carica del plasma sita all'interno della struttura cilindrica. Essendo la velocità della carica espressa dalla (23) si suppone che $\dot{r} = v$, $|W| \ll m_0 c^2$ e $|v| < c$. Oltre al termine classico in W', l'espressione della forza contiene allora un termine in W/r che rappresenta a seconda del segno della carica una forza correttiva, attrattiva o repulsiva, che può contribuire alla contrazione della corrente, cioè alla produzione del fenomeno di strizione (pinch-effect). Si segnala contemporaneamente anche il caso dei sistemi toroidali.

(*) *Traduzione a cura della Redazione.*

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Identification of Heavy Hypernuclei from K⁻ Capture by Primary Star Analysis (*).

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(ricevuto il 27 Febbraio 1961)

Summary. — Several hypernuclei of $A \geqslant 7$ have been uniquely identified from an analysis of the parent K^- capture reactions. This method has proved of great value in the choice of the correct identity when the decay process offered various alternative interpretations. An example of the decay ${}^{13}C_\Lambda \to \pi^- + {}^{13}N$, $B_\Lambda = (10.8 \pm 0.5)$ MeV has been thus identified for the first time. A second example of the decay ${}^{12}B_\Lambda \to \pi^- + 3\,{}^4He$, $B_\Lambda = (9.9 \pm 0.6)$ MeV is reported here, confirming the previous observation of this hypernuclide (¹). Some progress is made in resolving the composition of a group of heavy two-body decays. None of the events studied here is inconsistent with K^- capture on light nuclei (C, N, O); two events require a two-nucleon capture process.

1. – Introduction.

A recent study of mesic decays of hyperfragments produced by K^--reactions at rest in emulsion (¹) has yielded considerable data on the binding energies of H_Λ, He_Λ, Li_Λ, Be_Λ, and B_Λ. Most of the uniquely identified events were H_Λ and He_Λ, both because light mesic hyperfragment (HF) occur with greater

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(¹) R. Ammar, R. Levi-Setti, S. Limentani, W. E. Schlein, P. E. Slater and P. H. Steinberg: *Nuovo Cimento*, **15**, 181 (1960), hereafter referred to as I.

frequency and because the decays of heavier HF are more difficult to analyze unambiguously. Only 133 out of the 265 mesic HF decays detected in I could be uniquely identified on the basis of decay kinematics. The reasons for such losses are numerous, but generally the presence of one or more very short tracks (*i.e.* range $\leqslant 3\ \mu m$) in an event precluded its unique identification. In such cases, neither collinearity in two-prong events nor coplanarity in three-prong events could be reliably judged. As was shown in I, even in the most easily identified events, the emission of low energy neutrons ($\leqslant 0.5$ MeV) could not be experimentally excluded. This problen is expected to become increasingly serious in decays of heavy HF both because there are more neutrons which can be emitted, and because experimental errors allow the detection of these neutrons only above a higher threshold. Even if it were guaranteed that only charged particles are emitted, the range error in the shortest prong would frequently be so large that any of several nuclei within the estimated range limits could balance the momentum of the remaining prongs.

In such cases it is clear that additional information is needed. This information can come only from direct track measurements to determine the hyperfragment charge and mass (*e.g.* profile measurements) or else from an analysis of the production reaction. The latter type of analysis as a method for the identification of hypernuclei was introduced first by GILBERT *et al.* [2] and further used in two other instances [3,4] (*). The reactions studied in these three examples were:

$$K^- + {}^{12}C \to {}^5He_\Lambda + {}^4He + 2\,{}^1H + n + \pi^- + Q_a\,,$$

$$K^- + {}^{12}C \to {}^7Li_\Lambda + {}^4He + {}^1H + \pi^- + Q_b\,,$$

$$K^- + {}^{16}O \to {}^7Li_\Lambda + 2\,{}^4He + {}^1H + \pi^- + Q_c\,.$$

That these reactions involved ${}^{12}C$ and ${}^{16}O$ is an indication that K^- captures in light nuclei are important in the production of hyperfragments. Furthermore, such captures are easier to analyze than captures on heavy nuclei. It is now known that the emission energies of hypernuclei, other than H_Λ and He_Λ, from K^- and Σ^- absorptions are generally considerably less than the classical Coulomb barrier presented to « heavy » hypernuclei ($Z \geqslant 3$) by heavy

(2) F. C. GILBERT, C. E. VIOLET and R. S. WHITE: *Phys. Rev.*, **103**, 248 (1956).

(3) BANNIK, GULYATMOV, KOPILOVA, NOMOFILOV, PODGORETSKY, RACHIMBAYEV and USMANOVA: *High Energy Laboratory of the Joint Institute for Nuclear Research*; Preprint (1957) *USSR*.

(4) P. H. FOWLER: *Phil. Mag.*, **3**, 1460 (1958).

(*) *Note added in proof.* – A third instance has been reported by M. TAHER-ZADEH: *Nuovo Cimento*, **17**, 980 (1960). The reaction was $K^- + {}^{12}C \to {}^9Be_\Lambda + \pi^- + {}^3He$.

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emulsion nuclei (Ag and Br) [5]. It is reasonable to assume as a working hypothesis that hypernuclei of $Z \geqslant 3$ are produced almost exclusively in light emulsion nuclei. These nuclei and their abundances in emulsion are listed in Table I.

TABLE I. – *Abundance and isotopic purity of elements* C, N, O *in nuclear emulsion.*

Element	Abundance in emulsion [6] (no. atoms/cm³)	Natural isotopic abundance [7]
^{12}C	$1.389 \cdot 10^{22}$	98.89%
^{16}O	$0.937 \cdot 10^{22}$	99.76%
^{14}N	$0.318 \cdot 10^{22}$	99.64%

The EFINS events from I, and several previously unreported events, have been studied here for the purpose of investigating the extent to which primary star analysis would be generally useful in resolving certain existing ambiguities in the identification of hypernuclei heavier than He_Λ. Those events which offer new information are discussed here in some detail.

2. – Method.

Success of kinematic analysis of the production reaction depends on finding a single reaction for which the momentum is balanced. In addition, the total energy release should match the value predicted from the difference in masses between the initial and final configurations of particles. Such differences involve well-known nuclear masses and less well-known hypernuclear masses. The greatest uncertainty in the mass of a particular hypernucleus comes from the uncertainty in its binding energy (B_Λ). We can write for the production reaction:

$$\mathrm{K}^- + \text{nucleus} \to \mathrm{HF} + \text{particles} + Q' , \tag{1}$$

where

$$Q' = Q'_0 + B_\Lambda .$$

([5]) R. LEVI-SETTI: private communication.

([6]) M. SHAPIRO: *Handb. d. Phys.*, **45**, 342 (1958).

([7]) D. SROMIGER, J. M. HOLLANDER and G. T. SEABORG: *Rev. Mod. Phys.*, **30**, 585 (1958).

Here Q'_0 is known to better than $\pm .02$ MeV. Similarly for the hyperfragment decay:

$$\text{HF} \to \text{particles} + Q \, , \tag{2}$$

where

$$Q = Q_0 - B_\Lambda \, .$$

Here again, Q_0 depends on nuclear masses known to better than $\pm .02$ MeV. In a given event, several pairs of reactions, (1)–(2), may satisfy momentum balance, especially when neutrons are involved. However, the condition

$$Q + Q' = Q_0 + Q'_0 \tag{3}$$

is expected to hold for only one pair of production-decay reactions.

Condition (3) must be used in events where the object is to identify a new hypernuclide and measure its B_Λ. In other events, in which kinematic analysis of the hypernuclear decay has narrowed down the possibilities to only two or three, the object is to decide between two alternative interpretations; *e.g.* $^8\text{Li}_\Lambda \to \pi^- + 2\,^4\text{He}$ and $^9\text{Li}_\Lambda \to \pi^- + \text{n} + 2\,^4\text{He}$ for both of which hypernuclides B_Λ is already known from several unambiguous events [1]. Here Q' can be predicted with sufficient accuracy to make the agreement between measured and expected Q' a good criterion for identification.

Production reactions (1) can be classed as mesic or non-mesic, as is conventional for decay reactions (2). We have derived results from both types. However, just as π^- mesic decays are more tractable, so generally are π^- mesic production reactions, and for similar reasons:

1) Lower energy release.

2) More charged particles are emitted.

3) At least one particle (the pion) is easily identified.

In the analyses to be described, recent heavy ion range-energy data were used [8] to determine the HF emission energies. These data, determined for a wide range of ions, are expected to account adequately for the effects of electron pickup, and thus are especially useful for the short $Z > 3$ HF connecting tracks with which we are concerned.

All range measurements were corrected to standard emulsion density and the R-E data of BARKAS *et al.* [9] were used for parricles of $Z = 1$ and $Z = 2$.

(8) H. H. HECKMAN, B. L. PERKINS, W. G. SIMON, F. M. SMITH and W. H. BARKAS: UCRL-8763 (1959).

(9) W. H. BARKAS, P. H. BARRETT, P. CUER, H. HECKMAN, F. M. SMITH and H. K. TICHO: *Nuovo Cimento*, 8, 185 (1958).

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TABLE II. – *Primary star ranges and angles of events considered in this paper.*

Event	HF			Prong (1)			Prong (2)			Method of range determination	Prong (3) (*)		
	Range	θ	λ	Range	θ	λ	Range	θ	λ		Range (cm)	θ	λ
2-2K	8.7 μm	333.4°	+29.0°	47.4 μm	202.5°	+25.0°	387 μm	52.9°	+ 7.5°	Blob count	2.6 ± .5	214.7°	−31.2°
10-1K	12.1 μm	223.7°	+ 9.0°	11.6 μm	74.5°	+46.8°	—	—	—	—	—	—	—
13-6K	6.3 μm	140.4°	− 36.9°	15.58 mm	327.0°	+29.5°	4.79 mm	113.7°	+ 66.6°	Observed range	2.52 ± .07	303.3°	−30.2°
18-7K	74 μm	99.0°	+27.1°	84±8 mm	299.6°	0°	—	—	—	—		—	—
25-3Σ	26.6 μm	159.7°	+53.5°	77.4 μm	34.9°	−35.8°	—	—	—	—	—	—	—
B26-1K	29.5 μm	254.5°	+58.0°	321 μm	292.4°	− 52.8°	3[illegible].0 μm (**) 4.54 mm	125.0° 311.0°	+46.3° + 4.4°	Observed range	4.31 ± .08 (proton)	104.6°	−27.2°
B29 4Σ	6.3 μm	357.2°	−30.0°	13.6 μm	70.6°	−48.1°	—	—	—	—	—	—	—
NM30-1Σ	5.9 μm	—	−30.6°	—	—	—	—	—	—	—	—	—	—
38-3K	5.2 μm	275.8°	0°	569 μm	159.0°	+35.3°	42.8 μm	83.3°	−51.0°	Blob count	10.4 ±3.3	90.7°	−51.6°
B40-5K	34.4 μm	72.7°	+46.0°	14.2 μm	242.6°	− 21.7°	140.1 μm	222.8°	− 66.3°	Blob count	12.5 ±5.8	281.8°	−61.9°
B46-3K	13.1 μm	346.1°	0°	437 μm	180.0°	−18.0°	95 μm	242.3°	+11.7°	—	—	116.9°	+36.4°
46-7K	14.4 μm	286.2°	+28.8°	17.4 mm	119.5°	− 9.7°	98.6 μm	7.4°	+20.0°	Blob count	12.5 ±5.8	99.5°	−67.7°
50-3K	15.2 μm	294.8°	+40.5°	127.5 μm	67.1°	+ 1.8°	—	—	—	—	—	—	—
51-5K	4.2 μm	88.1°	0°	9.10 mm	293.9°	−51.6°	101 μm	310.5°	+46.6°	—	—	—	—
B56-2K	10.1 μm	153.0°	+ 8.9°	3.38 mm	286.4°	+ 8.0°	4.03 mm	348.0°	− 3.3°	Blob count	7.15 ± .42	76.0°	−45.5°
58-3K	4.0 μm	—	—	—	—	—	—	—	—	—	—	—	—

(*) Pion unless otherwise noted.
(**) This event has four prongs.

The vector momentum sum and the total energy release (Q') were then found for all permutations of possible (Z, A) assignments to the prongs involved. An IBM computer was used for these calculations [10]. The K^- exposure and emulsion stack from which these events were obtained is described adequately in I. It should be pointed out that the dimensions of the stack $(6 \times 10 \times 15)$ cm^3 are such that the charged pions from the primary HF production reactions did not in general come to rest in the stack (*i.e.*, the extrapolated ranges in emulsion can be as long as ~ 18 cm). Thus in the majority of events the sign of the pion could not be determined directly, and occasionally the uncertainty in its energy from blob counting was so great (say ± 20 MeV) that condition (3) was not very restrictive. When the direct determination of the pion energy was so poor, we replaced the requirement of momentum balance by the weaker requirement that the pion momentum vector should point opposite to the resultant momentum vector of the remaining particles. For an interpretation acceptable in this sense, the magnitude of the pion momentum was assumed equal to that of the vector momentum of the other particles, and Q' could be calculated. In these events we were forced to consider interpretations involving neutron emission, by assuming a value for the pion momentum which would give a lower limit for the energy release.

Table II contains the raw data (angles, ranges) for the primary star particles of the events considered in this paper.

3. – Primary analysis of two-body decays.

A frequently occurring π^--decay mode was the form $^A Z_\Lambda \to {}^A(Z+1) + \pi^-$ These decays are usually identified by the collinearity between the pion and the recoil, and by observation of the characteristic pion and recoil ranges. Fig. 1*a* shows a plot of recoil range *vs.* pion range for the two-body decays in the sample considered here (we henceforth refer to these as π-r events). Although angle measurements on short recoils cannot be good enough to accurately define colinearity, the two distinct clusterings of pion ranges for the events having recoil ranges $\lesssim 4\ \mu$m provide additional evidence that the decays are of the π-r class. The well-defined classes of $^3H_\Lambda$ and $^4H_\Lambda$ presented in I have been included in Fig. 1*a* for the sake of illustrating the natural spread of the clusterings. Fig. 1*b* shows in addition the expected positions of π-r decays of hypernuclides whose binding energies have not yet been determined. In plotting these we have assumed nominal binding energies deduced from interpolation of the B_Λ *vs.* A data in Fig. 2. It should be noted that

[10] F. W. Inman: UCRL Report No. 3815 (1957).

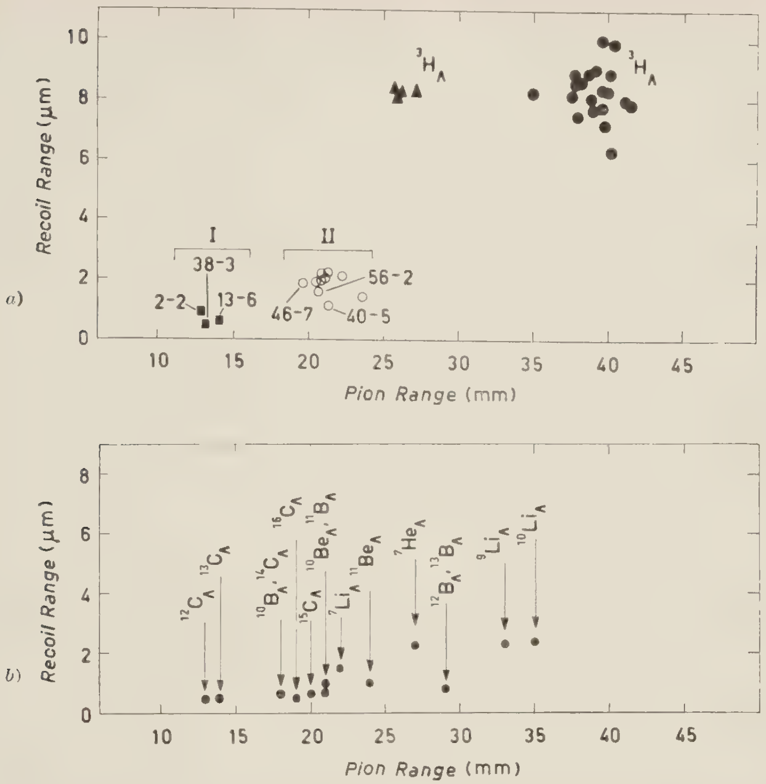

Fig. 1. – Recoil range *vs.* pion range for π-r events. Fig. 1*a* is reproduced from I, with the addition of Group I events. The events considered in the present analysis carry identification numbers. Fig. 1*b* shows the expected positions of π-r events for hypernuclides with $3 \leqslant Z \leqslant 6$.

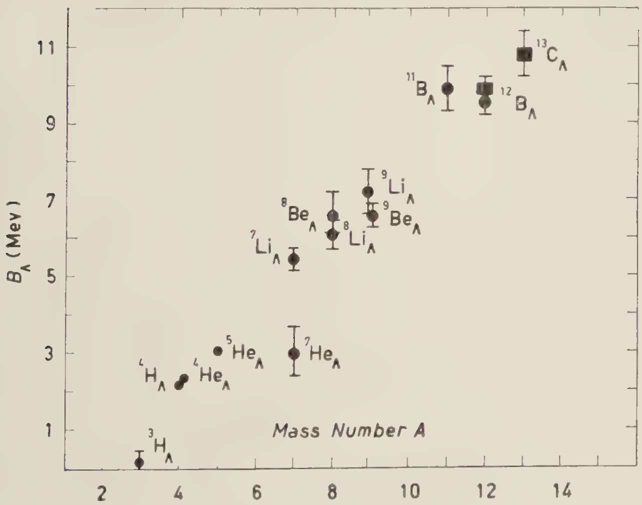

Fig. 2. – Binding energy (B_Λ) *vs.* mass number (A). Showing the new points presented here superimposed on the results of I. • reported in I; ▪ present work.

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deviations of ± 2 MeV from a linear B_Λ-*vs.*-A dependence are observed in I and indicate the degree of uncertainty in these interpolated values. These deviations are interpreted as due to spin dependence in the Λ-nucleon interaction and to differences in core structure. Furthermore, it is expected that the emission of particles in excited states will also lead to « apparent » binding energies different from the true ones.

Analyses of cluster I and three events in cluster II are contained in the following two subsections.

3·1. *Group I events* ($R_{\text{decay}\,\pi} \sim 14$ mm). – This class of π-r events has a π range of ~1.4 cm and a « collinear » recoil of ~1 μm range. In view of the information contained on Fig. 1*a*, 1*b*, we consider the 3 events of this group to be most probably examples of

$$^{12,13}\mathrm{C}_\Lambda \to \pi^- + {}^{12,13}\mathrm{N}\,.$$

A more cautious approach to the analysis, however, leads us to consider also possible identification as $^{10}\mathrm{B}_\Lambda$ and $^{14}\mathrm{C}_\Lambda$. $^{15,16}\mathrm{C}_\Lambda$ are immediately excluded by virtue of the fact that all production events are of the type: HF $+2$ prongs $+\pi$.

a) Event 13-6K (Prim.: HF+2H+π^-; Sec.: π^-+2) ([11]). This event is considered separately from the other two events in Group I because we were able to trace the primary pion to rest. The knowledge of the π^- range and energy leads to correspondingly smaller uncertainties in the measured Q-values and allows a unique identification to be made.

The results for this event are shown in Table III *a*, *b*. It is seen that we can make a unique interpretation of the event as an example of:

$$\mathrm{K}^- + {}^{16}\mathrm{O} \to {}^{13}\mathrm{C}_\Lambda + \pi^- + 2\,{}^1\mathrm{H} + \mathrm{n}\,.$$

The decay is therefore:

$$^{13}\mathrm{C}_\Lambda \to {}^{13}\mathrm{N} + \pi^-\,.$$

([11]) The notation used in the event identification number and in the primary and secondary star type is as follows: event 13-6K means event 13-6 was produced by a K^- meson. Some of the events discussed here were produced by a Σ^- in which cases the number would be, *e.g.* 25-3Σ. If a capital B precedes the event number as in *e.g.* B29-4Σ, the event was originally reported in the EFINS-NU collaboration Binding Energy paper (I). The primary and secondary star type contains whatever information could be learned from inspection of the event. Thus, (Prim.: HF+H+1+π; Sec.: π^-+r) means that the primary star consists of a hyperfragment, a charge one track, another track of unknown charge and a pion of unknown charge (unknown because it could not be traced to its end point); the secondary star consists of a π^- and a recoil.

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TABLE III *a*). – *Primary analyses of the group I events* (13-6, 2-2, and 38-3).

Reaction ($+Q'$ in MeV)		(MeV) Q_0+Q_0' 2-body decay	Experimental $(Q+Q')$ (MeV)			Experimental Q' (MeV) ΔP (MeV/c)		
			13.6	2-2	38-3	13-6	2-2	38-3
$K^-+{}^{16}O$	${}^{14}C_\Lambda+\pi^-+2{}^1H+160.0$	193.7	178.0 ± 2.5	89.3 ± 5.0	145.1 ± 20.9	149.3 ± 2.2 $111\ \pm 68$	62.0 ± 4.4 $471\ \pm 68$	117.5 ± 20.9 $239\ +87$
	${}^{13}C_\Lambda+\pi^-+{}^1H+{}^2H+156.3$	184.8	178.2 ± 3.5	89.9 ± 5.0	148.7 ± 22.0	149.5 ± 3.0 $273\ \pm 66$	62.6 ± 5.0 $442\ \pm 57$	121.0 ± 22.0 $219\ \pm 85$
	${}^{13}C_\Lambda+\pi+2{}^1H+n+154.1$	182.6	185.0 ± 10.5	192.2 ± 26.2	173.8 ± 31.7	156.3 ± 10.2 $114\ \pm 66$	164.9 ± 26.2 $451\ \pm 57$	146.1 ± 31.7 $232\ \pm 85$
	${}^{12}C_\Lambda+\pi^-+{}^1H+{}^2H+n+137$	164.7	230.2 ± 20.2	180.6 ± 24.1	172.0 ± 30.5	201.5 ± 20.2 $277\ \pm 64$	153.2 ± 24.1 $422\ \pm 55$	144.3 ± 30.5 $222\ \pm 82$
	${}^{12}C_\Lambda+\pi^-+2{}^1H+2n+135.6$	162.5	$>181.9\pm 4.5$	$>136.6\pm 15.0$	$>158.6\pm 22.0$	$>153.2\pm 4.0$ $118\ \pm 63$	$>109.2\pm 15.0$ $431\ +55$	$>130.9\pm 22.0$ $223\ \pm 82$
	${}^{10}B_\Lambda+\pi^-+{}^4He+{}^1H+n+144.0$	177.0	—	$155\ \pm 17.3$	188.3 ± 29.9	—	128.1 ± 17.3 $346\ \pm 45$	160.5 ± 29.9 $239\ \pm 69$
$K^-+{}^{14}N$	${}^{10}B_\Lambda+\pi^-+{}^2H+{}^1H+n+140$	173.8	238.8 ± 18.0	158.4 ± 17.6	163.3 ± 25.5	210.0 ± 18.0 $311\ \pm 53$	131.0 ± 17.6 $3\ 5\ \pm 45$	135.5 ± 25.5 $173\ \pm 68$
	${}^{10}B_\Lambda+\pi+2{}^1H+2n+138$	171.6	$>183.9\pm 5.5$	$>117.6\pm 12.0$	$>152.3\pm 24.0$	$>155.1\pm 5.1$ $164\ \pm 52$	$>90.2\pm 12.0$ $343\ \pm 44$	$>124.5\pm 24.0$ $174\ \pm 68$

TABLE III *b*). – *Decay analysis and possible binding energies* (*assuming 2-body decays*) *of the group I events* (13-6, 2-2 *and* 38-3).

Event number		13-6	2-2	38-3
Range secondary π^- (mm)		14.06	12.95	13.22
π^- energy (MeV)		28.32	26.97	27.30
Possible identification	Expected B_Λ (MeV) (inferred from Fig. 2)	Experimental B_Λ (MeV)		
$^{12}C_\Lambda$	($\sim$ 10)	9.15	10.50	10.17
$^{13}C_\Lambda$	($\sim$ 11)	10.85	12.20	11.87
$^{14}C_\Lambda$	($\sim$ 12)	16.47	17.82	17.49
$^{10}B_\Lambda$	($\sim$ 8)	12.62	13.97	13.63

Using the measured decay π^- range of 14.06 mm,

$$B_\Lambda = (10.8 \pm 0.5) \text{ MeV} .$$

This B_Λ value is plotted on Fig. 2, and is seen to form part of the consistent trend of B_Λ to increase at the rate of ~ 1 MeV/nucleon for the HF thus far observed. A discussion of this behavior has been given by DALITZ and DOWNS ([12]). An electron track, apparently associated with the very short recoil, is seen at its end. This is consistent with the knowledge that ^{13}N is a β^+ emitter.

b) Events 2-2*K* and 38-3*K* (Prim.: HF+H+r+π; Sec.; π^-+r). These two HF decays are quite similar to the preceding event, the decay π^- ranges being 12.95 mm and 13.22 mm respectively. Possible production reactions are listed in Table III*a*. In these two events the primary pions did not stop in the stack, and the errors in the *Q*-values for the production reactions are correspondingly larger.

The results of the production analysis given in Table III*a* allow us to exclude only the first two reactions. We note, however, that the secondary analysis given in Table III*b* leads to values of B_Λ ($^{10}B_\Lambda$) which depart by more than 5 MeV from the B_Λ-A data of Fig. 2. Thus, considering this as an unlikely departure, it seems possible to exclude $^{10}B_\Lambda$ and to conclude that these two events are most probably examples of

$$^{12,13}C_\Lambda \to {}^{12,13}N + \pi^- .$$

([12]) R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **111**, 967 (1958).

3'2. *Group II events* ($R_{\text{decay}\,\pi} \sim 22$ mm). – This class of π-r events has a π^- range of about 2 cm and a « collinear » recoil of 1-2 μm range. With the use of Fig. 1 we consider the following decay reactions as possible:

$$^{7}\text{Li}_{\Lambda} \rightarrow \pi^- + {}^{7}\text{Be}$$

$$^{10}\text{Be}_{\Lambda} \rightarrow \pi^- + {}^{10}\text{B}$$

$$^{11}\text{Be}_{\Lambda} \rightarrow \pi^- + {}^{11}\text{B}$$

$$^{10}\text{B}_{\Lambda} \rightarrow \pi^- + {}^{10}\text{C}$$

$$^{11}\text{B}_{\Lambda} \rightarrow \pi^- + {}^{11}\text{C}$$

$$^{14}\text{C}_{\Lambda} \rightarrow \pi^- + {}^{14}\text{N}$$

As described in Section **1** it is impossible to identify the short recoil nuclei from knowledge of range and momentum alone and therefore kinematic analysis of the decay stars cannot discriminate among the above reactions. Previously reported events of this type have been analyzed as $^{7}\text{Li}_{\Lambda} \rightarrow \pi^- + {}^{7}\text{Be}$. Two HF discussed below are shown to have a different probable identity while for a third event the $^{7}\text{Li}_{\Lambda}$ interpretation is confirmed.

a) Event B56-2K (Prim.: HF$+2\,{}^{1}$H; Sec.: $\pi^- + \text{r}$). This event was tentatively assigned in I to the group of possible $^{7}\text{Li}_{\Lambda} \rightarrow \pi^- + {}^{7}\text{Be}$. The energy of the primary pion was determined here from ionization to an accuracy of ± 3 MeV. The primary protons were identified by means of integral gap length *vs.* range. The results of production analysis show the event to be ambiguous, allowing either of the reactions:

Production	Decay	Q_0+Q_0'	$Q+Q'$	Δp
$K^- + {}^{12}\text{C} \rightarrow {}^{10}\text{Be}_{\Lambda} + 2\,{}^{1}\text{H} + \pi^-$	$^{10}\text{Be}_{\Lambda} \rightarrow {}^{10}\text{B} + \pi^-$	188.7	180 ± 4	44 ± 62
$K^- + {}^{14}\text{N} \rightarrow {}^{11}\text{B}_{\Lambda} + 2\,{}^{1}\text{H} + \text{n} + \pi^-$	$^{11}\text{B}_{\Lambda} \rightarrow {}^{11}\text{C} + \pi^-$	184.9	189 ± 10	91 ± 75

For the $^{10}\text{Be}_{\Lambda}$ reaction the experimental $Q+Q'$ deviates by two standard deviations from the expected $Q_0 + Q_0'$, thus making the alternative identification as $^{11}\text{B}_{\Lambda}$ more probable. The possible binding energies are shown in Table IV.

b) Event 46-7K (Prim.: HF$+{}^{1}$H$+$H(He); Sec.: $\pi^- + \text{r}$). The primary pion in this event was steep (68°) and left the stack. Thus it was impossible to obtain a reliable energy determination even from ionization measurements on the track. In the analysis we assumed a pion range of (12.5 ± 6) cm, con-

TABLE IV. – *Possible binding energy values for the group II events (assuming 2-body decays)* (56-2, 46-7, 40-5).

Event number		56-2	46-7	40-5
Range secondary π^- (mm)		20.56	19.66	21.33
π^- energy (MeV)		35.48	34.54	36.29
Possible identification	B_Λ (MeV) inferred from Fig. 2	Experimental B_Λ (MeV)		
$^7Li_\Lambda$	(5.5)	6.84	7.80	6.01
$^{10}Be_\Lambda$	($\sim$ 8)	8.08	9.03	7.26
$^{11}Be_\Lambda$	($\sim$ 9)	12.78	13.74	11.96
$^{10}B_\Lambda$	($\sim$ 8)	5.31	6.24	4.49
$^{11}B_\Lambda$	(9.9)	10.24	11.20	9.42
$^{14}C_\Lambda$	($\sim$ 12)	9.21	10.16	8.39

sistent with the grain count. The ambiguity in charge identification of one of the primary prongs results in a large number of reactions (25) which had to be tested.

The lack of knowledge of the primary pion range leads to a large number of ambiguities in the production analysis of which the two most probable reactions are the same as given above for event 56-2. $^{10}Be_\Lambda$ has $\Delta Q = 1.30\,\sigma$ and $^{11}B_\Lambda$ has $\Delta Q = 1.40\,\sigma$. $^7Li_\Lambda$ has $\Delta Q = 2.4\,\sigma$ and is correspondingly less probable $(\Delta Q = [(Q_0 + Q_0') - (Q + Q')])$.

c) Event B 40-5K (Prim.: HF$+{}^1$H$+1+\pi$; Sec.: π^-+r). This event was reported in I among the possible $^7Li_\Lambda \to \pi^- + {}^7Be$. Production analysis of this event leads to two possible identifications:

$$K^- + {}^{12}C \to {}^7Li_\Lambda + \pi^- + {}^1H + {}^4He \text{ (or } {}^3He + n) \qquad {}^7Li_\Lambda \to \pi^- + {}^7Be$$

$$K^- + {}^{16}O \to {}^{11}Be_\Lambda + \pi^+ + \text{nucleons} \qquad {}^{11}Be_\Lambda \to \pi^- + {}^{11}B\,.$$

We can argue against the $^{11}Be_\Lambda$ identification by noting that the HF range is 34.4 μm and that in I all HF heavier than Li_Λ were found to have ranges $\lesssim 10$ μm. Longer ranges can, however, arise from 2-nucleon K^- absorptions (see Section 7). However, in such cases, phase space considerations make the emission of a pion exceedingly improbable.

For the $^7Li_\Lambda$ production reaction we measure $Q = (134.8 \pm 35.2)$ MeV compared with the expected $Q' = 152.2$ MeV. $\Delta P = (34 \pm 46)$ MeV/c. The inferred pion space angles for the $^7Li_\Lambda$ production reaction differ from the

Fig. 3. – Event No. NM30-1Σ. Most probable interpretation: $K^- + p \to \Sigma^- + \pi^+$ (capture on bound proton); $\Sigma^- + {}^{12}C \to n + {}^{12}B_\Lambda$; ${}^{12}B_\Lambda \to {}^{8}Li + {}^{2}H + {}^{1}H + n$.

measured angles by the following amounts:

$$\text{azimuth:}\quad (7 \pm 5)^\circ\,,$$

$$\text{dip:}\quad (8 \pm 7)^\circ\,.$$

We conclude that the event is more probably $^7\mathrm{Li}_\Lambda$.

4. – $^{11}\mathrm{B}_\Lambda$ and $^{12}\mathrm{B}_\Lambda$ events.

a) Event NM 30-1Σ (Prim.: HF+0; Sec.: $^8\mathrm{L}(^8\mathrm{B})+{}^1\mathrm{H}+\mathrm{H}(\mathrm{He})+\mathrm{n}$'s). This event, while not yielding a precise B_Λ because it is a non-mesic decay, illustrates how a single preferred interpretation may be extracted from many *a priori* possibilities. The event is shown in Fig. 3. By virtue of the decay configuration of the HF, the minimum charge is 5, while a maximum charge of 7 is allowed by a capture on $^{16}\mathrm{O}$. The following arguments allow further discrimination against:

B_Λ: No HF heavier than $^{12}\mathrm{B}_\Lambda$ is possible. $^{11}\mathrm{B}_\Lambda$ is improbable because its only possible decay ($^{11}\mathrm{B}_\Lambda \to {}^8\mathrm{Li} + 2\,{}^1\mathrm{H} + \mathrm{n}$) with the configuration of this event leads to $B_\Lambda = (38 \pm 13)$ MeV. Lighter isotopes are excluded by the decay configuration.

C_Λ: No HF heavier than $^{14}\mathrm{C}_\Lambda$ is possible. Nothing lighter than $^{13}\mathrm{C}_\Lambda$ is allowed by the HF decay configuration. $^{14}\mathrm{C}_\Lambda$ is unfavored because the only possible production reaction:

$$\Sigma^- + {}^{14}\mathrm{N} \to {}^{14}\mathrm{C}_\Lambda + \mathrm{n} + 83\ \mathrm{MeV}$$

would produce a HF of range 4.8 μm compared with the observed (5.9 ± 1) μm. Furthermore, decay analysis (assuming emission of one neutron) leads to $B_\Lambda = -(28 \pm 20)$ MeV. $^{13}\mathrm{C}_\Lambda$ could have the observed configuration ($^{13}\mathrm{C}_\Lambda \to {}^8\mathrm{Li} + {}^3\mathrm{He} + {}^1\mathrm{H} + \mathrm{n}$) but leads to a $B_\Lambda = (-26 \pm 18)$ MeV.

N_Λ: The possible production reactions are:

		HF emission energy (MeV)
$\Sigma^- + {}^{16}\mathrm{O} \to$	$^{12}\mathrm{N}_\Lambda + 5\mathrm{n} + 23$ MeV	< 6.8
	$^{13}\mathrm{N}_\Lambda + 4\mathrm{n} + 37$ MeV	< 8.7
	$^{14}\mathrm{N}_\Lambda + 3\mathrm{n} + 58$ MeV	< 10.2
	$^{15}\mathrm{N}_\Lambda + 2\mathrm{n} + 70$ MeV	< 8.2
	$^{16}\mathrm{N}_\Lambda + \mathrm{n} + 81$ MeV	4.8

To obtain a HF range of 5.9 μm for any of these cases would require the HF emission energy to be ~ 9 MeV. It can be seen that even those cases which are not excluded in the above list are quite improbable since they require in each case that all the available neutrons have approximately the same vector momentum. Nothing further is learned from decay analysis using charge 7 for the HF.

We conclude that the event is most probably an example of

$$\Sigma^- + {}^{12}\mathrm{C} \to {}^{12}\mathrm{B}_\Lambda + \mathrm{n} + 74\ \mathrm{MeV}\,, \qquad {}^{12}\mathrm{B}_\Lambda \to {}^{8}\mathrm{Li} + {}^{1}\mathrm{H} + {}^{2}\mathrm{H}\ (\mathrm{or}\ {}^{1}\mathrm{H} + \mathrm{n}) + \mathrm{n}\,.$$

The expected ${}^{12}\mathrm{B}_\Lambda$ range is in this case 5.8 μm, which compares well with the observed range of (5.9 ± 1) μm. For the case in which only one free neutron is emitted in the decay, $B_\Lambda = (21 \pm 14)$ MeV is obtained. This is consistent with the data of Fig. 2.

b) Event B46-3K (Prim.: HF+He+1+π; Sec.: π^-+3, reported in I). In I the following two interpretations of the HF decay were presented:

$$(a) \qquad {}^{11}\mathrm{B}_\Lambda \to \pi^- + {}^{3}\mathrm{He} + 2\,{}^{4}\mathrm{He} \qquad B_\Lambda = (9.9 \pm 0.6)\ \mathrm{MeV}$$
$$\Delta p = (23 \pm 19)\ \mathrm{MeV}/c$$

$$(b) \qquad {}^{13}\mathrm{B}_\Lambda \to \pi^- + \mathrm{n} + 3\,{}^{4}\mathrm{He} \qquad B_\Lambda = (12.1 \pm 1.6)\ \mathrm{MeV}$$
$$p_n = (66 \pm 20)\ \mathrm{MeV}/c\,.$$

On the basis of the low value of Δp for reaction (*a*) it was concluded that this was the more likely interpretation.

In the primary star analysis, knowledge of $Z = 5$ for the HF and $Z = 2$ for one of the other primary prongs, reduces the possible production reactions to three:

	Measured energy release Q'_{meas} (MeV)
i) $K^- + {}^{16}\mathrm{O} \to {}^{11}\mathrm{B} + {}^{4}\mathrm{He} + {}^{1}\mathrm{H} + \pi^- + 154$ MeV	190 ± 35
ii) $K^- + {}^{16}\mathrm{O} \to {}^{11}\mathrm{B}_\Lambda + {}^{3}\mathrm{He} + {}^{1}\mathrm{H} + \pi^- + \mathrm{n} + 133$ MeV	$125 < Q' < 182$
iii) $K^- + {}^{18}\mathrm{O} \to {}^{13}\mathrm{B}_\Lambda + {}^{4}\mathrm{He} + {}^{1}\mathrm{H} + \pi^-$	

The natural abundance of heavy oxygen isotopes is very small ($< 0.2\,\%$) thus allowing us to exclude iii) as a possibility. For i) and ii), since the pion could not be traced to its ending point, the pion energy is inferred from the residual momentum of the other 3 prongs. In ii) the lower limit for Q_{exp} is determined by using a pion energy estimate from blob count and the « upper limit » is determined by using a null value for the neutron energy.

In conclusion, identification as $^{11}B_\Lambda$ given in I is confirmed, although the production reaction remains ambiguous.

c) Event 58-3K (Prim.: HF+0; Sec.: π^-+3). The primary star (HF+0) indicates that the total charge of the system (K$^-$+nucleus) is taken by the HF, making the HF have $Z=5$, 6 or 7. No $Z=6$ or 7 interpretations gave acceptable results in the secondary analysis and only two $Z=5$ interpretations were possible ($^{12}B_\Lambda$ and $^{13}B_\Lambda$). $^{13}B_\Lambda$ is ruled out for lack of nucleons in the production process. The range of the HF, $(4\pm1)\,\mu m$, is consistent with the range ($3.08\,\mu m$) expected from

$$K^- + {}^{12}C \to {}^{12}B_\Lambda + \pi^0\,, \qquad {}^{12}B_\Lambda \to \pi^- + 3\,{}^4He\,.$$

The B_Λ ($^{12}B_\Lambda$) is calculated to be (9.90 ± 0.6) MeV. This is consistent with the first example of $^{12}B_\Lambda$ ($B_\Lambda = 9.6\pm0.6$) MeV reported in I.

5. – $^{8,9}Li_\Lambda$ events.

Discrimitation between the reactions: $^8Li_\Lambda \to 2\,{}^4He+\pi^-$ and $^9Li_\Lambda \to 2\,{}^4He + n + \pi^-$ is impossible if the neutron in the $^9Li_\Lambda$ decay has a momentum $\leqslant (30\div40)$ MeV/c (the range of momentum uncertainty in 3-body coplanar decays [1]). Fig. 4 shows the energy release plotted *vs.* the residual momentum

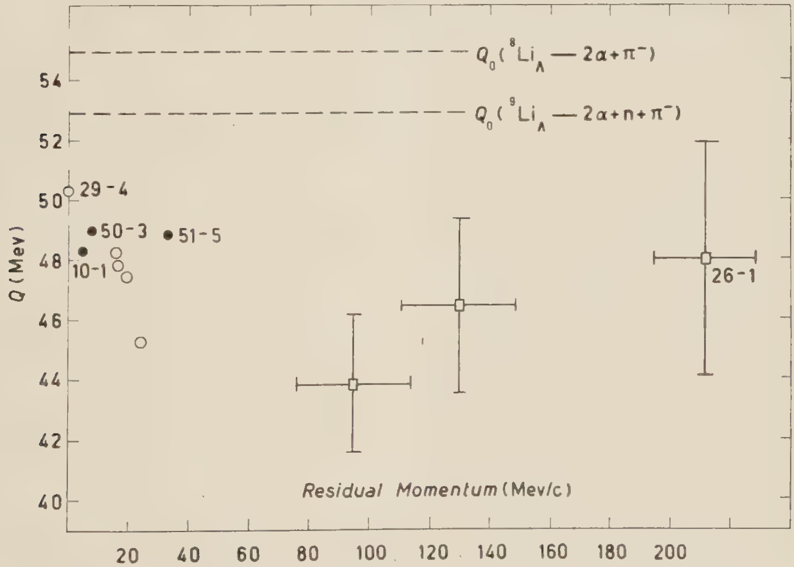

Fig. 4. – Residual momentum *vs.* visible energy release for events identified as $^8Li_\Lambda \to 2\alpha+\pi^-$ or $^9Li_\Lambda \to 2\alpha+n+\pi^-$. The points labeled by identification number refer to events discussed in the text. In the notation of Section 2, $B_\Lambda = Q_0 - Q$ where Q is the visible energy release (plus the neutron energy in the case of $^9Li_\Lambda$). ○ $^8Li_\Lambda$ reported in I; □ $^9Li_\Lambda$ reported in I; ● New Li_Λ events.

in the decays of the events reported in I and of three additional events. Both the ${}^8\mathrm{Li}_\Lambda$ and ${}^9\mathrm{Li}_\Lambda$ decays are characterized by their high energy releases (relative to other hypernuclear decays) due to the formation of two α-particles. We show here that in some cases production analysis permits us to state a preference between these two possibilities.

a) Events 10-1K and 50-3K (Prim.: HF+1; Sec.: π^-+2 (coplanar)). Both of these events exhibit decays which lead to their identification as ${}^8\mathrm{Li}_\Lambda$ (no evidence of neutron emission, $\Delta p < 10$ MeV/c). In each case a second visible prong is produced in the production process. There are only three primary reactions which are possible.

		Q'_{meas} (MeV)	
		10-1	50-3
i)	$K^-+{}^{12}C \to {}^8\mathrm{Li}_\Lambda+{}^3\mathrm{He}+n+\pi^0+143$ MeV	$33 < Q' < 175$	$52 < Q' < 234$
ii)	$K^-+{}^{12}C \to {}^8\mathrm{Li}_\Lambda+{}^4\mathrm{He}+\pi^0+163$ MeV	176 ± 30	261 ± 25
iii)	$K^-+{}^{12}C \to {}^9\mathrm{Li}_\Lambda+{}^3\mathrm{He}+\pi^0+146$ MeV	194 ± 31	247 ± 26

The « upper » and « lower » limits indicated for Q in the first reaction are determined by attributing the residual momentum to the π^0 and n respectively —there is of course no actual upper limit to the measured Q in these cases.

Only reaction i) is possible for event 50-3 while for event 10-1, neither ii) or iii) can be ruled out. Thus we confirm that event 50-3 is an example of the reaction:

$$K^- + {}^{12}C \to {}^8\mathrm{Li}_\Lambda + {}^3\mathrm{He} + n + \pi^0\,, \qquad B_\Lambda = (6.1 \pm 1.0)\ \mathrm{MeV}$$

while 10-1 is more probably

$$K^- + {}^{12}C \to {}^8\mathrm{Li}_\Lambda + {}^4\mathrm{He}\ (\text{or } {}^3\mathrm{He} + n) + \pi^0\,, \qquad B_\Lambda = (6.6 \pm 1.0)\ \mathrm{MeV}\,.$$

b) Event B29-4Σ (Prim.: HF+1; Sec.: π^-+2 (coplanar)). This event was identified in I as ${}^8\mathrm{Li}_\Lambda \to 2\,{}^4\mathrm{He} + \pi^-$ ($B_\Lambda = (4.8 \pm 1.0)$ MeV). It is similar to event 10-1 and 50-3 except that it is produced by Σ^- absorption. The possible primary reactions are:

		Q'_{meas} (MeV)
i)	$\Sigma^-+{}^{12}C \to {}^8\mathrm{Li}_\Lambda+{}^3\mathrm{He}+2n+41$ MeV	$> 33\pm6$
ii)	$\Sigma^-+{}^{12}C \to {}^8\mathrm{Li}_\Lambda+{}^4\mathrm{He}+n+61$ MeV	66 ± 13
iii)	$\Sigma^-+{}^{12}C \to {}^9\mathrm{Li}_\Lambda+{}^3\mathrm{He}+n+44$ MeV	63 ± 13

Both i) and ii) are acceptable. iii) has a Q'_{meas} about 1.5 σ too high. Thus the event is more likely $^8\text{Li}_\Lambda$.

c) Event 51-5K (Prim.: HF+^{1}H+1; Sec.: π^-+2 (coplanar)). As can be seen in Fig. 4, this event has the highest residual momentum of the group. Its B_Λ if $^8\text{Li}_\Lambda$ is (6.2 ± 1.0) MeV, if $^9\text{Li}_\Lambda$ is (3.5 ± 1.5) MeV.

The following primary reactions are possible.

	Q'_{meas} (MeV)
$K^- + {}^{12}C \to {}^8\text{Li}_\Lambda + {}^1H + {}^3H + \pi^0 + 143.6$ MeV	118.4 ± 15.2
$K^- + {}^{14}N \to {}^8\text{Li}_\Lambda + {}^1H + {}^4He + \pi^0 + n + 150.5$ MeV	153.7 ± 17.0 (assuming n at rest)
$K^- + {}^{12}C \to {}^9\text{Li}_\Lambda + {}^1H + {}^2H + \pi^0 + 140.4$ MeV	127.4 ± 16.5
$K^- + {}^{14}N \to {}^9\text{Li}_\Lambda + {}^1H + {}^4He + \pi^0 + 153.6$ MeV	147.0 ± 17.0

Thus we see that primary analysis furnishes no decisive information for the identification of this event. If we consider the capture in ^{12}C as being more probable then we have for

$$^8\text{Li}_\Lambda\,, \quad \Delta Q = 1.6\,\sigma \quad \text{and for} \quad {}^9\text{Li}_\Lambda\,, \quad \Delta Q = 0.8\,\sigma\,.$$

6. – Evidence for a new decay mode of $^7\text{Li}_\Lambda$ or $^8\text{Li}_\Lambda$.

a) Event 25-3Σ (Prim.: HF+1; Sec.: π^-+r, (apparently colinear)). The HF decay shows a π^- of range 9.92 mm and a very short recoil ($\sim 0.5\ \mu$m). The 77 μm prong which is associated with the HF production reaction can be $Z=1$ or $Z=2$. For these two possibilities charge conservation tells us that the primary reaction must be one of the following:

$$\Sigma^- + {}^{12}C \to \left\{ \begin{array}{l} \text{Be}_\Lambda + \text{H} + (\text{n's}) \\ \text{Li}_\Lambda + \text{He} + (\text{n's}) \end{array} \right.$$

$$\Sigma^- + {}^{14}N \to \left\{ \begin{array}{l} \text{B}_\Lambda \ + \text{H} + (\text{n's}) \\ \text{Be}_\Lambda + \text{He} + (\text{n's}) \end{array} \right.$$

$$\Sigma^- + {}^{16}O \to \left\{ \begin{array}{l} \text{C}_\Lambda \ + \text{H} + (\text{n's}) \\ \text{B}_\Lambda \ + \text{He} + (\text{n's})\,. \end{array} \right.$$

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In almost every case (assuming all possible HF identifications), Q'_{meas} is sufficiently higher than the expected energy release that the reaction can be ruled out. Some typical examples are:

		Q'_{meas} (MeV)
$\Sigma^- + {}^{14}\text{N} \to$	${}^{11}\text{B}_\Lambda + {}^1\text{H} + 3\text{n} + 49.8$ MeV	> 85.8
$\Sigma^- + {}^{16}\text{O} \to$	${}^{12}\text{C}_\Lambda + {}^1\text{H} + 4\text{n} + (\sim)\ 37$ MeV	> 91.5
$\Sigma^- + {}^{14}\text{N} \to$	${}^{10}\text{Be}_\Lambda + {}^3\text{He} + 2\text{n} + (\sim)\ 50$ MeV	> 73.3
	${}^{10}\text{Be}_\Lambda + {}^4\text{He} + \text{n} + (\sim)\ 70$ MeV	111.7
$\Sigma^- + {}^{16}\text{O} \to$	${}^{12}\text{B}_\Lambda + {}^3\text{He} + 2\text{n} + 46.0$ MeV	> 115.4
	${}^{12}\text{B}_\Lambda + {}^4\text{He} + \text{n} + 66.6$ MeV	184.1

The uncertainties in Q'_{meas} are less than 10 MeV.

The only acceptable reactions are:

		Q'_{meas} (MeV)
$\Sigma^- + {}^{12}\text{C} \to$	${}^8\text{Li}_\Lambda + {}^3\text{He} + 2\text{n} + 40.8$ MeV	$> 44.7 \pm 2.4$
	${}^8\text{Li}_\Lambda + {}^4\text{He} + \text{n} + 61.4$ MeV	65.6 ± 4.8
	${}^7\text{Li}_\Lambda + {}^3\text{He} + 3\text{n} + 32.9$ MeV	$> 37.4 \pm 1.6$
	${}^7\text{Li}_\Lambda + {}^4\text{He} + 2\text{n} + 53.5$ MeV	$> 44.8 \pm 2.4$

If the HF is ${}^7\text{Li}_\Lambda$ its decay must be of the form:

$$ {}^7\text{Li}_\Lambda \to \pi^- + {}^6\text{Be} + \text{n}\,, \qquad B_\Lambda = (4.9 \pm 1.0)\ \text{MeV} $$

while if it is ${}^8\text{Li}_\Lambda$, the decay could be

$$ {}^8\text{Li}_\Lambda \to \pi^- + {}^7\text{Be} + \text{n}\,, \qquad B_\Lambda = (8.9 \pm 1.0)\ \text{MeV}\,. $$

The decay mode ${}^8\text{Li}_\Lambda \to \pi^- + {}^6\text{Be} + 2\,\text{n}$ leads to $B_\Lambda < -0.5$ MeV and thus can be excluded.

The recoil (be it ${}^7\text{Be}$ or ${}^6\text{Be}$) can be emitted in an excited state and thus these B_Λ's may be upper limits in this sense.

7. – Hypernuclei produced in 2-nucleon absorptions.

a) Event 18-7K (Prim.: HF $+ {}^1$H (*no* π^0); Sec.: $\pi^- + 3$). The secondary analysis of this event gives results ambiguous between ${}^8\text{Be}_\Lambda$ and ${}^9\text{Be}_\Lambda$. The

primary star is extremely unusual in that the HF has a range of 74 μm (momentum 770 MeV/c) compared with the usual $5 \div 10$ μm range for Be_Λ hypernuclei. A proton of range (84 ± 8) mm (momentum 605 MeV/c) is also emitted from the production reaction. This corresponds to an energy of 213 MeV thereby kinematically forbidding the emission of a pion from the reaction. We thus believe these particles to have come from the two-nucleon primary reaction:

$$K^- + 2p \to \Lambda + p + 317 \text{ MeV}.$$

The visible charge from the primary reaction is 5, thus indicating that the event may have taken place in ^{12}C (it should be noted that because of the high emission energy of the HF, the *a priori* assumption of a production in C, N, O is much less substantiated in this case). With this assumption, however the possible reactions and the Q'_{meas}'s are:

	$Q'_{meas.}$ (MeV)
$K^- + {}^{12}C \to {}^{8}Be_\Lambda + {}^{1}H + 3n + 269$ MeV	> 244
$K^- + {}^{12}C \to {}^{9}Be_\Lambda + {}^{1}H + 2n + 288$ MeV	> 262

We see that the primary analysis offers no further discrimination between $^{8,9}Be_\Lambda$ in this case.

b) Event B26-1K (Prim.: $HF + {}^1H + 2H + 1$ (*no* π^0); Sec.: $\pi^- + 2$ (non-coplanar)). Analysis of the HF decay star (reported in I) yielded a unique interpretation ${}^9Li_\Lambda \to \pi^- + 2\,{}^4He + n$. Inspection of the primary star and assumption of a C, N, O absorption led us to conclude that the HF was produced in one of the following reactions:

	$Q'_{meas.}$ (MeV)	ΔP (MeV/c)	
$K^- + {}^{16}O \to {}^9Li_\Lambda + {}^3H + {}^2H + 2\,{}^1H + 253$ MeV	249 ± 2	24 ± 30	(1)
$K^- + {}^{16}O \to {}^9Li_\Lambda + 3\,{}^2H + {}^1H + 249$ MeV	258 ± 3	119 ± 30	(2)
$K^- + {}^{16}O \to {}^9Li_\Lambda + 2\,{}^2H + 2\,{}^1H + n + 243$ MeV	250 ± 4	—	(3)
$K^- + {}^{16}O \to {}^9Li_\Lambda + {}^2H + 3\,{}^1H + 2n + 240$ MeV	$> 247 \pm 2$	—	(4)
$K^- + {}^{16}O \to {}^9Li_\Lambda + {}^3H + 3\,{}^1H + n + 247$ MeV	248 ± 2	—	(5)
$K^- + {}^{16}O \to {}^9Li_\Lambda + 4\,{}^1H + 3n + 238$ MeV	$> 247 \pm 2$	—	(6)

In view of the large visible energy release it can be concluded that a π^0 is not emitted in the production process. Reaction (5) seems the most probable.

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One proton emitted in the production process has an energy of 122 MeV. This strongly suggests that the primary K^- absorption process was of the type:

$$K^- + p + p \rightarrow \Lambda + p$$

or

$$K^- + n + p \rightarrow \Sigma^- + p$$

followed by:

$$\Sigma^- + p \rightarrow \Lambda + n\,.$$

8. – Discussion.

The present analysis has furnished the first unique example of $^{13}C_\Lambda$, the second example of $^{12}B_\Lambda$, and has confirmed the tentative identification in I of event B46-3K as $^{11}B_\Lambda$. The new $(B_\Lambda\text{-}A)$ points determined here are shown along with the results of I in Fig. 2. This analysis has further resolved several members of the two groups of ambiguous heavy events which exhibit π-r decays and some of the $^{8,9}Li_\Lambda$ events. A summary of the event identifications is given in Table V.

TABLE V. – *Summary of event identifications resulting from the production analyses given here.*

Event	Original ID if included in I	New unique or confirmed ID	Probable ID	Alternate ID
13-6K	—	$^{13}C_\Lambda$	—	—
2-2K and 38-3K	—	$^{12,13}C_\Lambda$	—	—
B56-2K	$^{7}Li_\Lambda$	—	$^{11}B_\Lambda$	$^{10}Be_\Lambda$
46-7K	—	—	$^{11}B_\Lambda$ or $^{10}Be_\Lambda$	—
B40-5K	$^{7}Li_\Lambda$	$^{7}Li_\Lambda$	—	—
NM30-1Σ	—	$^{12}B_\Lambda$	—	—
B46-3K	$^{11}B_\Lambda$	$^{11}B_\Lambda$	—	—
58-3K	—	$^{12}B_\Lambda$	—	—
10-1K	—	—	$^{8}Li_\Lambda$	$^{9}Li_\Lambda$
50-3K	—	$^{8}Li_\Lambda$	—	—
B29-4Σ	$^{8}Li_\Lambda$	—	$^{8}Li_\Lambda$	$^{9}Li_\Lambda$
51-5K	—	—	$^{9}Li_\Lambda$	$^{8}Li_\Lambda$
25-3Σ	—	$^{7,8}Li_\Lambda$	—	—
18-7K	—	$^{8,9}Be_\Lambda$	—	—
B26-1K	$^{9}Li_\Lambda$	$^{9}Li_\Lambda$	—	—

The observation of two hyperfragments produced as a result of multi-nucleon K^- capture has some bearing on the understanding of hyperfragment production processes. Events 18-7K and B26-1K are two observed examples of a fast pick-up process, resulting as they do from fast Λ's produced in $K^- + 2n \to \Lambda + n$.

Although many of the hyperfragment production stars analysed here gave inconclusive results, it is nevertheless important to note that none was inconsistent with production in a light element (C, N, O). If heavy hyperfragment production in heavy elements were a very probable process, one would expect the production stars to be distinguished by the absorption of large momenta and small energies by heavy nuclear recoils, or else by the disintegration of these nuclei. There was no evidence for either characteristic.

The most massive hyperfragments one could hope to produce in normal emulsion [13] with slow K^- have $A = 16$:

$$K^- + {}^{16}O \to \left|\begin{array}{ll} \pi^- + {}^{16}O_\Lambda & \quad (\text{HF range} \sim 1.6\ \mu\text{m})\,. \\ \pi^0 + {}^{16}N_\Lambda & \\ \pi^+ + {}^{16}C_\Lambda & \end{array}\right.$$

The unique ranges of the charged primary pions would allow ${}^{16}O_\Lambda$ and ${}^{16}C_\Lambda$ to be detected. Detailed HF studies using the approach adopted here with emulsion stacks large enough to stop the primary pion will allow one to obtain further information concerning the Λ-nuclear interaction.

* * *

It is a pleasure to thank Professors R. LEVI-SETTI and V. L. TELEDGI for their continuous advice and assistance in the course of this work. We are particularly indebted to Professor TELEGDI for having suggested this investigation and to Professor LEVI-SETTI for remeasurement of several data and for aid in the preparation of the manuscript. We are grateful to Professor W. H. BARKAS for providing us with heavy ion range-energy data prior to publication. This experiment could not have been carried out without the co-operation of Dr. E. J. LOFGREN and Dr. H. H. HECKMAN, and the bevatron crew.

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(13) Note, however, that R. LEVI-SETTI and W. E. SLATER (*Nuovo Cimento*, **14**, 895 1959) have suggested the use of K^- induced fission in Uranium loaded emulsion as a possible means of producing heavy hypernuclei.

RIASSUNTO (*)

Da un'analisi delle reazioni di cattura di K^-, che li generano, abbiamo identificato in modo univoco parecchi ipernuclei con $A \geqslant 7$. Questo metodo si è dimostrato particolarmente valido per la scelta della identificazione corretta, quando il processo di decadimento offriva varie interpretazioni alternative. Un esempio di decadimento ${}^{13}C_\Lambda \to \pi^- + {}^{13}N$, $B_\Lambda = (10.8 \pm 0.5)$ MeV, è stato così per la prima volta identificato. Un secondo esempio del decadimento ${}^{12}B_\Lambda \to \pi^- + 3\,{}^4He$, $B_\Lambda = (9.9 \pm 0.6)$ MeV, viene qui riferito, confermando una precedente osservazione di questo ipernuclide (1). Si fa qualche progresso nel risolvere la composizione di un gruppo di decadimenti pesanti a due corpi. Nessuno degli eventi qui studiati è incompatibile con la cattura di un K^- da parte di un nucleo leggero (C, N, O); due eventi richiedono un processo di cattura a due nucleoni.

(*) *Traduzione a cura della Redazione.*

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On the Binding Energies of Mesic Hypernuclei (*).

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(ricevuto il 13 Marzo 1961)

Summary. — A detailed analysis of 76 mesic decays of hypernuclei, found in a G-5 emulsion stack exposed to stopping K^--meson beam at Berkeley is reported here. The binding energies of the known decay modes are in good agreement with the previously reported values. A value for the binding energy of $^3H_\Lambda$ is given based on data reported here and in the literature. Some new and rare decay modes have been observed and discussed.

1. – Introduction.

Reported here are the results of an experiment to determine binding energies of hypernuclei. The results presented here were obtained from eighty-nine mesic decays of hypernuclei produced by K^- capture in nuclear emulsion. These events were found in a single area scan in which 13 290 K^- capture stars were examined. A preliminary report of these results has been given elsewhere (1).

2. – Experimental details.

2·1. *Exposure and stack calibration.* – An emulsion stack (**) was exposed in December, 1957, to the 300 MeV/c enriched K^- beam at the Bevatron.

(*) Research supported by the United States Atomic Energy Commission.
(1) Y. Prakash, P. H. Steinberg, D. A. Chandler and R. J. Prem: *Bull. Am. Phys. Soc.*, 6, 40 (1961).
(**) We are greatly indebted to Dr. M. M. Shapiro and the Nuclear Emulsion group of the U.S. Naval Research Laboratory for allowing us the use of the stack for this experiment.

The stack consisted of 149 Ilford G-5 emulsion pellicles of dimensions $10\text{ cm}\times 15\text{ cm}\times 0.06\text{ cm}$. An estimated $2\cdot 10^4$ K^--mesons entered the stack along with a background of about 800 minimum ionizing particles per K^-.

The emulsion density was determined directly by successive weighings in air and methyl salycilate. Based on data from seven pellicles, the density was found to be $(3.807 \pm 0.002)\text{ g cm}^{-3}$. Original pellicle thicknesses were measured at specified locations throughout tha stack. During the course of the experiment a bi-monthly determination of the mean shrinkage factor was obtained from values measured at these locations. The mean shrinkage factor was observed to vary by no more than 2% during this time. In any given determination, the spread of individual shrinkage factors about the mean was also about 2%.

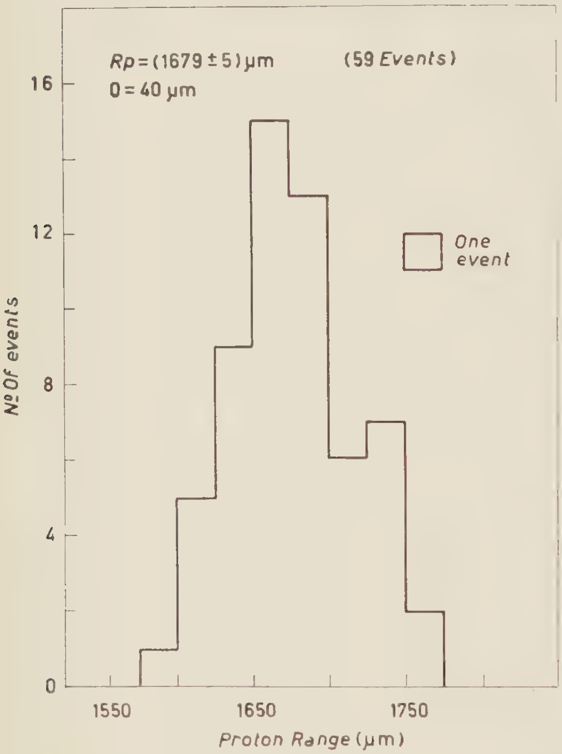

Fig. 1. – Histogram distribution of proton ranges from the decays at rest of Σ^+.

Mesic hypernuclei were found among the secondary tracks from K^- capture stars. These secondaries were followed until they stopped or left the plate and their endings were examined for associated events. Ranges and angles of the decay prongs of mesic hypernuclei were determined from at least two independent measurements ([2]).

An independent determination of the density was obtained indirectly from the mean proton range from fifty-nine Σ^+-decays. These events were identified on the basis of $3\times$(st. dev.) selection criteria on the proton range. In Fig. 1, the distribution of proton ranges of these events is shown. The mean of this distribution, $(1679 \pm 5)\ \mu\text{m}$,

([2]) P. H. Steinberg: *Ph. D. Thesis* (Northwestern University, Evanston, 1959). A detailed discussion is given on the measurement techniques and the assignment of measurement errors.

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was compared to the value $(1678 \pm 3)\,\mu\text{m}$ [3-4] for emulsion of standard density $3.815\,\text{g cm}^{-3}$. Then, in accordance with BARKAS *et al.* [5], the density was inferred to be $(3.810 \pm .013)\,\text{g cm}^{-3}$. Using the directly measured density, the mean proton range was reduced to its value in standard emulsion. This value was found to be $(1677 \pm 5)\,\mu\text{m}$. In addition, a Fry-White [6] analysis was made on these events as a check on the shrinkage factor assignment. The correction to the shrinkage factor thus obtained was found to be no larger than the experimental error.

2'2. *Analysis.* – The analysis of an event is an attempt to find a single set of particle assignments to the decay prongs such that:

1) The vector sum of momenta is zero within experimental error, and

2) The binding energy is positive [*] and consistent with the known binding energy of the assigned species, at least for the well-known hypernuclides.

For events assigned to hypernuclides of which there are ten or more known examples [7-11] we required that the calculated binding energy be within $3\delta B_\Lambda$ of the known mean. δB_Λ is the random binding energy error of that event. Events satisfying these criteria are said to be unique. Some decays required the assumption of a neutron, as for example, a non-colinear two-body decay or a non-coplanar three-body decay. Such events are said to be unique if only one identification leads to a positive and consistent binding energy. No attempt was made to assign more than one neutron to a decay.

For purpose of analysis the events were sub-divided into the following three groups.

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(3) J. MASON, W. H. BARKAS, J. N. DYER, H. H. HECKMAN, N. A. NICKOLS and F. M. M. SMITH: *Bull. Am. Phys. Soc.*, **5**, 224 (1960).

(4) J. BOGDANOWICZ, M. DANYSZ, A. FILIPKOWSKI, E. MARQUIT, E. SKRZYPCZAK, A. WROBLEWSKI and J. ZAKRZEWSKI: *Acta Phys. Pol.*, **19**, 277 (1960).

(5) W. H. BARKAS, P. H. BARETT, P. CUER, H. HECKMAN, F. M. SMITH and H. K. TICHO: *Nuovo Cimento*, **8**, 185 (1958).

(6) W. F. FRY and G. R. WHITE: *Phys. Rev.*, **90**, 207 (1953).

(*) The requirement that the binding energy be positive was relaxed for ${}^3\text{H}_\Lambda$ where the binding energy is close to zero and is smaller than the experimental error.

(7) R. LEVI-SETTI, W. E. SLATER and V. L. TELEGDI: *Suppl. Nuovo Cimento*, **10**, 68 (1958).

(8) R. AMMAR, R. LEVI-SETTI, W. E. SLATER, S. LIMENTANI, P. E. SCHLEIN and P. H. STEINBERG: *Nuovo Cimento*, **15**, 181 (1960).

(9) S. LOKANATHAN, D. K. ROBINSON and S. J. ST. LORANT: *Proc. Roy. Soc.*, A **254**, 470 (1960).

(10) J. SACTON: *Nuovo Cimento*, **15**, 110 (1960).

(11) J. TIETGE: private communication (1960).

a) Two-body decays of the type, ${}^{A}Z_{\Lambda} \to \pi^- + {}^{A}(Z+1)$. These will be referred to henceforth as π-r decays.

b) Three-body decays of the type, ${}^{A}Z_{\Lambda} \to \pi^- + p + {}^{A-1}Z$, to be referred to henceforth as π-p-r decays.

c) Others, including cases of neutron emission.

a) The only decays of this type observed in this experiment are:

$$ {}^{3}H_{\Lambda} \to \pi^- + {}^{3}He\,, $$
$$ {}^{4}H_{\Lambda} \to \pi^- + {}^{4}He\,. $$

Such decays are readily identifiable by the unique pion ranges, 26.4 mm and 39.5 mm for ${}^{3}H_{\Lambda}$ and ${}^{4}H_{\Lambda}$ respectively.

b) The method described by SLATER [12] was used in the analysis of events of this type. In this method the recoil is identified by comparing its range (R_0) and momentum (P_0), as determined by momentum balance against the proton and pion, to the adopted range *vs.* momentum curves. The recoil range R_0 is determined by the well measured projected length (R_p) and the dip angle as inferred from momentum balance.

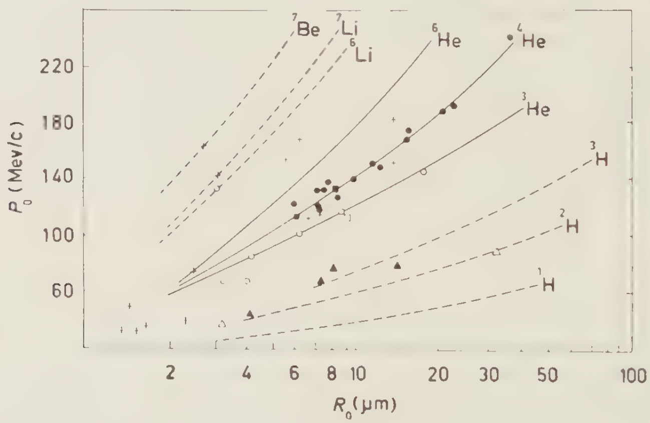

Fig. 2. – Range *vs.* momentum curves for light nuclei: ● ${}^{4}He$ from ${}^{5}He_{\Lambda}$(π-p-r), ○ ${}^{3}He$ from ${}^{4}He_{\Lambda}$(π-p-r), ▲ ${}^{3}He$ from ${}^{4}He_{\Lambda}$(π-p-e), △ ${}^{3}He$ from ${}^{4}He_{\Lambda}$(π-p-e), □ ${}^{3}He$ from ${}^{3}He_{\Lambda}$(π-p-r), ■ ${}^{4}He$ normalized point from six π-r decays of ${}^{4}He_{\Lambda}$, + non-uniquely identified π-p-r events, ——— normalized Wilkins curves, – – – Barkas curves.

The adopted range *vs.* momentum curves are shown in Fig. 2. The WILKINS [13] curve for ${}^{4}He$ was displaced toward lower ranges by 0.5 μm to fit the well determined $R_0 - P_0$ point of our six π-r decays of ${}^{4}H_{\Lambda}$. Curves for

([12]) W. E. SLATER: *Suppl. Nuovo Cimento*, **10**, 1 (1958).
([13]) J. J. WILKINS: A.E.R.E. Harwell, G/R 664 (1951).

other helium isotopes were derived from the adjusted ^{4}He curve. The lithium and beryllium curves were calculated [14] from range-energy data for heavy ions in emulsion given by HECKMAN *et al.* [15]. Curves for hydrogen isotopes were taken directly from the Barkas range-energy relation [16].

It is impossible to uniquely identify a recoil track for ranges less than 3 μm. In this region the separation in range of the various curves for a given charge is less than range measurement errors. In some cases where the recoil dip angle is large, unique identification is impossible even for larger ranges.

The discrepancy ($\Delta\lambda$) between the recoil dip angle as inferred from momentum balance and its measured value was noted for each event. The distribution of $\Delta\lambda$ for various R_0 intervals as shown in Fig. 3, provides experimental

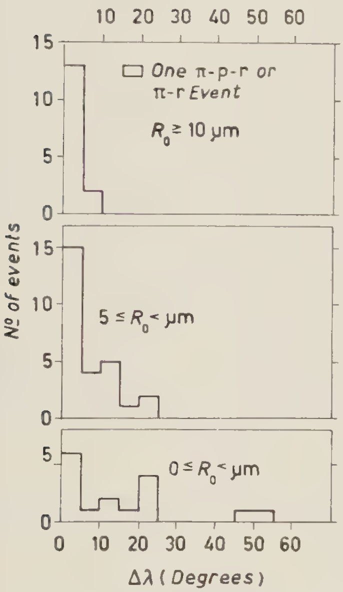

Fig. 3. – Histogram distribution of $\Delta\lambda$.

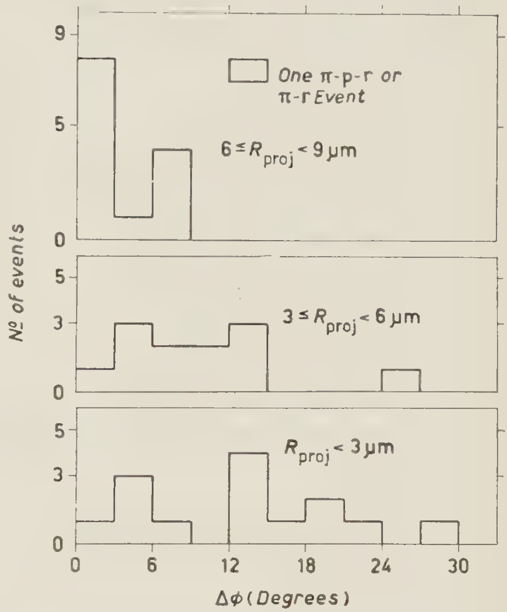

Fig. 4. – Histogram distribution of $\Delta\varphi$.

limits for establishing coplanarity for a given event. Histograms of $\Delta\varphi$, the analogous quantity for projected angles, are shown in Fig. 4 for various intervals of projected length. Distributions of both these quantities are useful as a guide in the assignment of measurement errors.

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(14) P. E. SCHLEIN and W. E. SLATER: private communication (1960).

(15) H. HECKMAN, B. L. PERKINS, W. G. SIMON, F. M. SMITH and W. H. BARKAS: U.C.R.L. 8763 (June, 1959).

(16) W. H. BARKAS: *Nuovo Cimento*, 8, 201 (1958).

c) The more complicated decays required the use of a digital computer for their analyses. The computer was programmed [17,18] to calculate a binding energy (B_Λ) and a momentum unbalance (ΔP) for all permutations among the isotopes of the assumed Z values for each decay prong. In addition the computer calculated the errors in B_Λ and ΔP as propagated from the assigned measurement errors [2].

Permissible identifications were selected from the output on the basis of ΔP being zero within its calculated error and B_Λ being positive and consistent. A single neutron was assumed only if a preliminary analysis failed to yield at least one permissible identification.

An extension of the method used for π-p-r decays was sometimes employed with advantage on four-body events (with no neutrons) for which the analysis produced several competing identifications. If the ambiguity was in the identification of the shortest and least measurable prong, data on remaining prongs were used to calculate its momentum and direction. The corrected range and momentum, thus obtained, was used to identify this prong. If, on the other hand, the ambiguity was in the identification of one of the longer prongs, the momentum direction for the short prong was computed for each identity of this prong. The comparison of the various sets of inferred angles (as computed for each prong identity) and the measured values was used as the basis for selecting a most highly favored decay scheme.

2·3. *Conventions and standards.* – The binding energy $B_\Lambda(Z, A)$ is conveniently expressed as the difference $(Q_0 - Q)$, where Q is the total kinetic energy release and Q_0 is the rest mass energy of the Λ and the core nucleus in its ground state (*) minus the sum of the rest mass energies of the decay products.

The mass of the Λ, proton, neutron and π^- used in this experiment are taken from the mass table of BARKAS and ROSENFELD [19]. The corresponding value of Q_Λ, the energy release in the charged decay mode of the free Λ, is (37.56 ± 0.13) MeV. This differs very little from the value (37.58 ± 0.15) MeV adopted in the EFINS-NU collaboration experiment [8]. Isotopic masses were taken from MATTAUCH [20].

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(17) F. W. INMAN: U.C.R.L. 3815 (1957).

(18) E. M. SILVERSTEIN: *Suppl. Nuovo Cimento*, **10**, 41 (1958).

(*) For unbound core nuclei the lowest energy state is used, *e.g.*, the core nucleus of ${}^9\mathrm{Be}_\Lambda$ is taken as the ground state of ${}^8\mathrm{Be}$ which is unstable against ${}^4\mathrm{He}$ emission by 96 keV.

(19) W. H. BARKAS and A. H. ROSENFELD: *Rochester Conference on High Energy Physics* (1960), p. 877.

(20) J. MATTAUCH, L. WALDMANN, R. BIERI and F. EVERLING: *Ann. Rev. Nucl. Sci.*, **6**, 179 (1956).

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The Barkas range-energy relation [16] was exclusively used for determining energies of singly charged particles. Before converting them to energy, ranges of pions were reduced to their values in standard emulsion by a formula given by SLATER [12].

2·4. *Sources of error.* – The principal systematic errors present in the determination of Q are listed as follows:

a) Uncertainty in the range-energy relation [16], $\delta_{RE} = 0.1$ MeV.

b) Uncertainty in the shrinkage factor, $\delta_S = 0.1$ MeV.

c) Uncertainty in the density correction for ranges, $\delta_D = 0.02$ MeV.

The dominant error in Q_0 is the uncertainty in Q_Λ. Errors in the isotopic masses amount to no more than 0.01 MeV and can be neglected. In so far as Q_Λ has been predominantly determined by range measurements in emulsion, its δ_{RE} is expected to be of the same magnitude and direction as the δ_{RE} in Q. As a result, this error should cancel in forming the difference, $B_\Lambda = Q_0 - Q$. The remaining error in Q_Λ, after its δ_{RE} has been extracted is 0.08 MeV [19]. Since this is in no way correlated with δ_D and δ_S in Q, the total systematic error propagated to B_Λ is 0.13 MeV.

The random error in Q for any event, not involving a neutron, is due mainly to the 3 % range straggling of the pion track. For such events, then, the random error is given by $0.018\,Q$ [2]. A direct measure of this error is found in the standard deviation of the B_Λ distribution for our eighteen $^5He_\Lambda$ events. This standard deviation is found to be 0.67 MeV, which is in good agreement with the value 0.60 MeV obtained from the mean Q of 33.5 MeV for $^5He_\Lambda$.

3. – Results.

Of a total of eighty-nine mesic hypernuclear decays found in this experiment, seventy-six could be completely measured (*i.e.*, the pion track could be traced to its end). Of the incomplete events, two pions interacted in flight while the others left the stack before coming to rest. In the analysis, forty-eight of the complete events were uniquely identified while the remaining twenty-eight were ambiguous between two or more competing decay schemes.

3·1. *Uniquely identified events.* – A breakdown of the forty-eight unique events into the various observed decay modes is given in Table I. One must note the relative frequencies of the various modes given here are not a measure of their natural relative production rates. This is due to the varying degrees of difficulty in analyzing the different modes.

TABLE I. – *Decay modes of uniquely identified mesic decays.*

Decay mode	$\bar{B}_\Lambda$ (MeV)	Number of events
$^3H_\Lambda \to \pi^- + {}^3He$	0.0 ± 0.8	1
$\to \pi^- + {}^1H + {}^2H$	-0.5 ± 0.3	4
$^4H_\Lambda \to \pi^- + {}^4He$	2.1 ± 0.4	6
$\to \pi^- + {}^1H + {}^3H$	2.1 ± 0.3	5
$\to \pi^- + {}^3He + n$	1.3 ± 0.7	1
$^4He_\Lambda \to \pi^- + {}^1H + {}^3He$	2.4 ± 0.2	8
$^5He_\Lambda \to \pi^- + {}^1H + {}^4He$	3.04 ± 0.15	17
$\to \pi^- + {}^2H + {}^3He$	2.9 ± 0.6	1 (a) (*)
$^7He_\Lambda \to \pi^- + {}^3H + {}^4He$	3.0 ± 0.8	1 (b)
$^7Li_\Lambda \to \pi^- + {}^1H + {}^2H + {}^4He$	5.5 ± 0.5	1 (b)
$^8Li_\Lambda \to \pi^- + 2\,{}^4He$	7.1 ± 0.9	1
$\to \pi^- + {}^4He + {}^3He + n$	4.1 ± 3.1	1 (a)
$^9Be_\Lambda \to \pi^- + {}^1H + 2\,{}^4He$	6.4 ± 0.5	1

(*) See note added in proof.
(*a*) Not previously reported.
(*b*) One example previously reported.

The binding energies (weighted means) derived from these events are presented in Table II. These binding energies are in good agreement with previously reported values [7-11]. In view of the fact that in this experiment $\bar{B}_\Lambda$

TABLE II. – *Binding energies of hypernuclei from uniquely identified mesic decays.*

Hypernucleus	$\bar{B}_\Lambda$ (a) (MeV)	$\delta\bar{B}_\Lambda$ (b) (MeV)	Number of events
$^3H_\Lambda$	-0.4	0.3	5
$^4H_\Lambda$	2.01	0.23	12
$^4He_\Lambda$	2.38	0.24	8
$^5He_\Lambda$	3.10	0.13	18
$^7He_\Lambda$	3.0	0.8	1
$^7Li_\Lambda$	5.5	0.5	1
$^8Li_\Lambda$	6.9	0.8	2
$^9Be_\Lambda$	6.4	0.5	1

(*a*) Values of $\bar{B}_\Lambda$ are based on $Q_\Lambda = (37.56 \mp 0.13)$ MeV.
(*b*) Errors quoted here do not include a small systematic error of 0.13 MeV.

for $^3H_\Lambda$ is negative, we felt it appropriate to review the present experimental status of this hypernucleus. A compilation of available data for this binding energy is given in Table III. One observes that this binding energy (0.04 ± 0.17) MeV, is positive even with the inclusion of the value reported here. In the compilation in Table III, all binding energies have been re-evaluated in accordance with the conventions described in Section 2·3.

TABLE III. – *Survey of binding energy data for* $^3H_\Lambda$.

Source	Number of events	$\bar{B}_\Lambda$ [a] (MeV)	$\delta\bar{B}_\Lambda$ [a] (MeV)
EFINS-NU collaboration and EFINS Survey [8]	13	0.10	0.26
LOKNATHAN, *et al.* [9]	4	0.6	0.3
VAISENBERG and SMIRNITSKII [21]	1	−0.6	0.7
Present work	5	−0.4	0.3
Weighted mean	23	0.04	0.17

(*a*) Binding energies and their errors have been re-evaluated for this compilation according to the conventions described in the text.

Several decay modes are reported here for the first time. Event 1-115-1, the first observed decay of $^5He_\Lambda$ not going through the π-p-r channel (*), decays according to the scheme

$$^5He_\Lambda \to \pi^- + {}^2H + {}^3He\,. \tag{1}$$

The rarity of this mode is in part a consequence of its relatively small decay energy which for this event is (15.9 ± 0.3) MeV. The other, Event 1-62-1, decays according to the scheme

$$^8Li_\Lambda \to \pi^- + {}^4He + {}^3He + n\,. \tag{2}$$

This also represents a departure from the dominant, more energetic decay

$$^8Li_\Lambda \to \pi^- + 2\,{}^4He\,.$$

Particle data and information pertaining to the analysis will be given in the Appendix for these and other events of special interest.

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(21) A. O. VAISENBERG and V. A. SMIRNITSKII: *Žurn. Ėksp. Teor. Fiz.*, **32**, 736 (1957).

(*) See note added in proof.

In addition to these, two events are reported here for the second time. Event 1-121-1 which confirms a previous observation [22,7], decays as

$$(3) \qquad {}^{7}\mathrm{Li}_{\Lambda} \to \pi^{-} + {}^{1}\mathrm{H} + {}^{2}\mathrm{H} + {}^{4}\mathrm{He}\,.$$

The second example of ${}^{7}\mathrm{He}_{\Lambda}$, Event 1-124-1, decays by the scheme

$$(4) \qquad {}^{7}\mathrm{He}_{\Lambda} \to \pi^{-} + {}^{4}\mathrm{He} + {}^{3}\mathrm{H}\,.$$

This is also the decay scheme of the first observed ${}^{7}\mathrm{He}_{\Lambda}$ event, reported by the EFINS-NU collaboration [8].

An event which analysed as a ${}^{3}\mathrm{H}_{\Lambda}$, π-r decay, but was rejected because of its anomalous binding energy ($B_{\Lambda} = 10.8$ MeV), bears special mention. A somewhat similar event was reported by VARFOLOMEEV *et al.* [23], who suggest the possibility that the apparent energy loss is due to the emission of a γ-ray. An anomalous π-r decay of ${}^{4}\mathrm{H}_{\Lambda}$, with $B_{\Lambda} = 6.1$ MeV, was observed by the EFINS-NU collaboration [8]. It seems quite improbable that the radiative decay hypothesis could account for all three cases. Of at least comparable probability is the possibility that a neutron is emitted in the direction of the pion. The *a priori* probability of this interpretation, based on solid angle consideration, is about $2\cdot10^{-3}$. On this basis, we obtain a number of fits to heavier hypernuclei. This event is also interpretable as a heavy π-p-r decay with an invisible recoil. The length of the hyperfragment track ($\sim 9\ \mu\mathrm{m}$) would seem to substantiate this latter view [24].

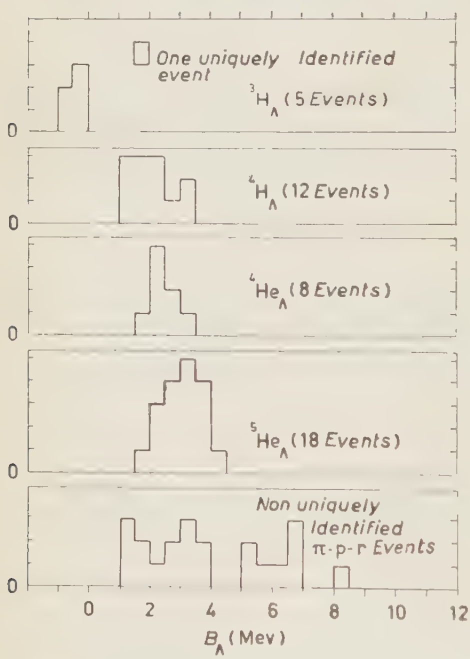

Fig. 5. – Histogram plots of binding energy ($\bar{B}_{\Lambda}$) for the abundant hypernuclear species.

[22] O. SKJEGGESTAD and S. O. SORENSEN: *Phys. Rev.*, **104**, 511 (1956).

[23] A. A. VARFOLOMEEV, R. I. GERASIMOVA and L. A. KARPOVA: *Dokl. Akad. Nauk SSSR*, **110**, 758 (1956).

[24] R. AMMAR, R. LEVI-SETTI, W. E. SLATER, S. LIMENTANI, P. E. SCHLEIN and P. H. STEINBERG: *Nuovo Cimento*, **19**, 20 (1961). Given here are range distributions for hypernuclei produced by K^{-} capture.

One decay in flight of a ${}^4H_\Lambda$ (π-p-r) was observed. The velocity at the point of decay was $\beta = 0.04$, corresponding to a flight time of $0.9 \cdot 10^{-12}$ s.

Binding energy histograms of individual events for the most abundant hypernuclides are shown in Fig. 5.

3'2. *Non-uniquely identified events.* – Twenty-two of the twenty-eight ambiguous events are of the π-p-r type. In most of these cases the ambiguity could be reduced to at most two identifications, usually of the same charge. The range-momentum curves for helium and hydrogen isotopes are sufficiently separated so that it is possible to differentiate between $Z = 1$ and $Z = 2$ down to a recoil momentum of 40 MeV/c. One can also make the distinction between π-p-r decays of heavy ($A > 5$) or light ($A < 5$) hypernuclei on the basis of the observed energy release. In Fig. 5 the binding energy distribution is shown for all the ambiguous π-p-r decays. The binding energy assigned to each event in this plot is an average over the various recoil identifications for that event.

4. – Discussion.

The dependence of B_Λ on hypernuclear mass number A is shown in Fig. 6. Principal features of the variation of B_Λ with A, as exhibited in Fig. 6, have

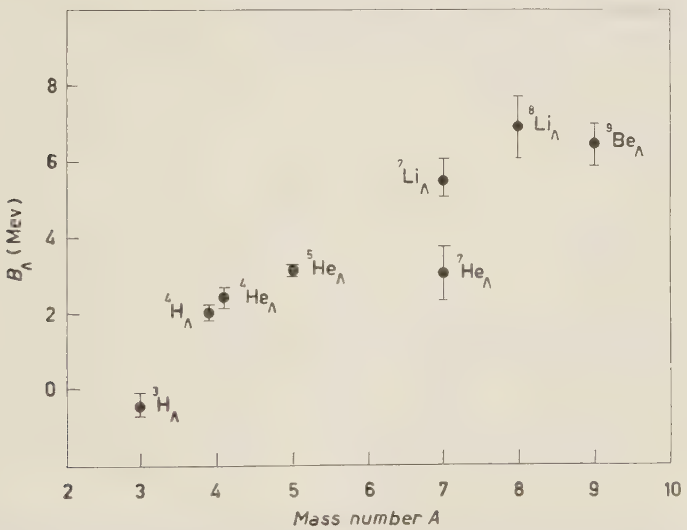

Fig. 6. – Distribution of Binding energy ($\bar{B}_\Lambda$) *vs.* mass number (A) for the uniquely identified mesic decays.

already been noted in the literature. One of these features is the tendency of B_Λ to increase with increasing values of A. Departures from a simple linear

relation between B_Λ and A are also exhibited, especially in the region $A < 5$.

The isotopic spin multiplet structure of hypernuclei is given further confirmation in this work. For example, the equality within experimental errors of the binding energies of ${}^4H_\Lambda$ and ${}^4He_\Lambda$ suggest that these hypernuclei are members of an isotopic spin doublet. This, along with isotopic spin assignment $T = 0$ for the Λ, is seen as evidence for charge symmetry of the Λ-nucleon interaction ([25]).

The failure to observe more than one hypernuclide of $A = 3$ and $A = 5$ suggests that ${}^3H_\Lambda$ and ${}^5He_\Lambda$ are isotopic spin singlets.

There now appears to be a real difference in the binding energies of the two $A = 7$ hypernuclides. With the observation of a second example of ${}^7He_\Lambda$ (reported here), the B_Λ of this hypernucleus is now fairly well established at about 3 MeV. The binding energy of ${}^7Li_\Lambda$, on the other hand, is about 5.5 MeV. This suggests that these hypernuclides have different isotopic spins, which is consistent with the fact that their core nuclei, ${}^6He$ and ${}^6Li$ have $T = 1$ and $T = 0$ respectively in their ground states (*).

* * *

The authors are highly grateful to Professor G. A. Snow for continuing criticism, helpful discussions, and enlighting suggestions throughout the work. The authors wish to thank the whole team of scanners, particularly Mr. P. Hesse, Mr. D. Garrison, Mr. L. Goodwyn, Mr. R. Ziepolt, Mr. J. Franklin, Mr. I. Haque, Miss R. Huang, and Mrs. R. Ouyang.

Appendix

The details of the analysis of the following events are described here.

(1) ${}^5He_\Lambda \to \pi^- + {}^2H + {}^3He$ (Event 1-115-1),

(2) ${}^8Li_\Lambda \to \pi^- + {}^4He + {}^3He + n$ (Event 1-62-1),

(3) ${}^7Li_\Lambda \to \pi^- + {}^1H + {}^2H + {}^4He$ (Event 1-121-1),

(4) ${}^7He_\Lambda \to \pi^- + {}^4He + {}^3H$ (Event 1-124-1).

1. *Event* 1-115-1. – Particle data for this event are given in Table IV.

The heavy particles in this decay where identified by comparing their ranges with the momenta obtained from a graphical construction based on

([25]) R. H. Dalitz: *Phys. Rev.*, **99**, 1475 (1955).

(*) The situation with $A = 7$ hypernuclei is clearly displayed in Fig. 7 of ref. ([8]).

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the pion momentum and the angles in the decay plane [2]. The momentum and range of the ^{3}He prong as inferred from momentum balance against the π^- and deuteron is $P_0 = 94$ MeV/c and $R_0 = 5.0$ μm. Coplanarity of the decay prongs was well established within experimental errors ($\Delta\lambda = 1°$ for the ^{3}He prong). The low energy release in this decay ((15.9 ± 0.3) MeV), excludes all modes but (1).

TABLE IV. – *Particle data for event* 1-115-1.

Prong	Range (μm)	Dip angle (°)	Proj. angle (°)
π^-	2340	7.5	0
^{2}H	90	−32	215
^{3}He	4.9	39	69
Hyperfragment	8.5	—	—

2. *Event* 1-62-1. Particle data for this event are given in Table V.

A preliminary computer analysis was made assuming charges $Z = 2$, 3 and 4 for both of the heavy decay prongs. The only acceptable decay scheme was (2). The existance of a neutron was established through a comparison between the momentum unbalance, $\Delta P = 86$ MeV/c and its calculated error, 29 MeV/c.

TABLE V. – *Particle data for event* 1-62-1.

Prong	Range (μm)	Dip angle (°)	Proj. angle (°)
π^-	9925	−49.4	0
^{3}He	6.5	28	23
^{4}He	4.8	−8	161
Hyperfragment	2.4	—	—

3. *Event* 1-121-1. – Particle data for this event are given in Table VI. The computer analysis yielded two possible identifications for this event.

(*a*) $$^7\text{Li}_\Lambda \to \pi^- + {}^2\text{H} + {}^4\text{He} + {}^1\text{H}; \quad B_\Lambda = (5.5 \pm 0.5)\ \text{MeV}, \quad \Delta P = (37 \pm 24)\ \text{MeV}/c\,;$$

(*b*) $$^8\text{Li}_\Lambda \to \pi^- + {}^3\text{H} + {}^4\text{He} + {}^1\text{H}; \quad B_\Lambda = (4.3 \pm 0.5)\ \text{MeV}, \quad \Delta P = (42 \pm 24)\ \text{MeV}/c\,.$$

Scheme (*a*) is chosen over (*b*) because the binding energy of (*b*) is inconsistent with the known value for $^8\text{Li}_\Lambda$.

TABLE VI. – *Particle data for event* 1-121-1.

Prong	Range (μm)	Dip angle (°)	Proj. angle (°)
π^-	10 581	7.0	0
^{2}H	17.7	−72	143
^{4}He	6.7	60	169
^{1}H	85	−38	239
Hyperfragment	31	—	—

4. *Event* 1-124-1. – Particle data for this event are given in Table VII. Preliminary identifications of the heavy particles in this decay were obtained from a graphical construction as in Event 1-115-1. The longer prong was identified as ^{1}H, ^{2}H, and ^{3}H while the shorter prong was identified as ^{3}He and ^{4}He. The only combinations of these, giving a positive B_Λ is scheme (4). The range and momentum of the ^{1}He, inferred from momentum balance against the pion and triton was $R_0 = 13.8$ μm and $P_0 = 170$ MeV/c ($\Delta\lambda = 6°$).

TABLE VII. – *Particle data for event* 1-124-1.

Prong	Range (μm)	Dip angle (°)	Proj. angle (°)
π^-	21 413	−53.6	0
^{3}H	21.3	+11	323
^{4}He	14.4	+30	156
Hyperfragment	127	—	—

Note added in proof.

A previous example of the decay $^5\mathrm{He}_\Lambda \to \pi^- + {}^2\mathrm{H} + {}^3\mathrm{He}$ has been reported by the EFINS-NU Collaboration. See Table IV: *Nuovo Cimento*, **19**, 20 (1961).

RIASSUNTO (*)

Riportiamo qui una dettagliata analisi di 76 decadimenti mesici di ipernuclei, riscontrati in un pacco di emulsioni G-5 esposta a Berkeley ad un fascio di mesoni K⁻ all'arresto. Le energie di legame dei modi di decadimento conosciuti sono in buon accordo con i valori pubblicati precedentemente. Diamo un valore per l'energia di legame del $^3\mathrm{H}_\Lambda$, basato sui dati riportati qui e nella bibliografia. Abbiamo osservato e discutiamo alcuni modi di decadimento nuovi e rari.

(*) *Traduzione a cura della Redazione.*

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Analytic Behavior of the Scattering Amplitude at Zero Energy (*).

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(ricevuto il 20 Marzo 1961)

Summary. — It is shown that the scattering amplitude has a two-sheeted branch point at zero kinetic energy. The analytic continuation into the second sheet is discussed in detail.

1. – Introduction.

In this paper we consider the scattering of identical particles of the mass m, spin and charge zero. For simplicity we exclude stable bound states of the system and any coupling to other particles. We further assume invariance under the transformation $A(x) \to - A(x)$ of the field operator (pair theory), *i.e.* particles can only be created in pairs.

The complete analytic behavior of the scattering amplitude $T(s, t)$ (s square of the center of mass energy, t negative square of the momentum transfer) at zero kinetic energy $s = 4m^2$ is determined by

i) the unitarity condition in the elastic scattering region;

(*) A short report on this work has been published in the *Proc. of the Rochester Conference* (1960), p. 351. Several of the results have been obtained independently in ref. (1-3).

(**) On leave of absence from Institut für theor. Physik der Universität, Hamburg.

(1) R. Blankenbecler: *Proc. of the Rochester Conference* (1960), p. 247; R. Blankenbecler, M. L. Goldberger, S. W. McDowell and S. B. Treiman: preprint (1961).

(2) J. G. Taylor: *Proc. of the Rochester Conference* (1960), p. 245; J. Gunson and J. G. Taylor: to be published.

(3) R. Oehme: preprint (1960).

ii) the analytic properties of the scattering amplitude on the first sheet. Following Symanzik's program [4] we will use this information for separating the two particle singularities from the scattering amplitude. It will be shown that $T(s,t)$ has a branch point at $s = 4m^2$ of the type $\sqrt{s-4m^2}$ and can be continued on the second sheet of $\sqrt{s-4m^2}$.

Unfortunately the theory of the analytic properties ii) on the first sheet is at present in a very fragmentary state. The Mandelstam representation of the scattering amplitude would provide complete information on ii) but has been proved in some order of perturbation theory only. For the equal mass case, however, MANDELSTAM [5] could obtain some analyticity of $T(s,t)$ in both variables without using perturbation theory. Using dispersion relations and regularity in the Lehmann ellipse MANDELSTAM proved that the scattering amplitude can be continued at least into the domain

$$|st(4m^2-s-t)| < A\,, \tag{1.1}$$

where the constant A is

$$A = 7168\, m^6$$

in case of a pair theory.

Since the results become much clearer we will assume the analytic properties of the Mandelstam representation throughout this paper. For a detailed account of the rigorous results on the basis of (1) we refer to the Appendix. In will be proved there that for a finite region surrounding the point $s = 4m^2$, $t = 0$ the analytic behavior of $T(s,t)$ can be obtained by using (1) only. The boundary of this region, however, is affected in a rather artificial manner by the particular shape of the domain (1) and has certainly no physical meaning.

In Section **2** the analytic behavior of the partial wave amplitude at zero kinetic energy is studied. Within the convergence domain of the partial wave expansions this leads to the complete analytic behavior of the scattering amplitude on the Riemann surface of $\sqrt{s-4m^2}$ (Section **3**). Further analytic continuation (*i.e.* beyond the convergence domain of the partial wave expansions) is obtained by using the (elastic) unitarity condition as the defining integral equation for the scattering amplitude on the second sheet (Section **4**). It will turn out that on the second sheet additional singularities appear, as for instance kinematical cuts, poles and the complex Landau singularities.

In Section **6** the analytic continuation of the K-matrix element from the elastic region to complex values of s and t is discussed by using Heitler's

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(4) SYMANZIK has developed a general method of analysing the many-particle structure of the retarded Green's functions, see K. SYMANZIK: *Journ. Math. Phys.*, **1**, 249 (1960).

(5) S. MANDELSTAM: *Nuovo Cimento*, **15**, 658 (1960).

integral equation. The K-matrix element is regular at $s=4m^2$, but has the same type of complex singularities as the scattering amplitude on the second sheet.

Finally the singularities at $s=4m^2$, $t=4m^2$ and $u=4m^2$ are treated simultaneously in a manner satisfying crossing symmetry (Section 7).

Though we restrict ourselves to the case of identical spinless particles the extension to other models is possible. If the scattered particles have different masses the kinematical cuts must be modified in the manner described by MacDowell (6). Multichannel situations can easily be treated by using the formulation of Bjorken (7) and Nauenberg (7). Utilizing methods developed by H. Joos the formalism can be extended to particles with spin (8).

We will use the following notations. Let k_1, k_2 denote the momenta of two incoming, k_1', k_2' the momenta of two outgoing particles. The scattering amplitude

$$\delta(k_1+k_2-k_1'-k_2')\,T(s,t) = T(k_1k_2|k_1'k_2') = -i(\Phi^{\text{in}}_{k_1k_2},\,(S-1)\,\Phi^{\text{in}}_{k_1'k_2'}) \tag{1.2}$$

is a function of

$$s=-(k_1+k_2)^2 \qquad \text{and} \qquad t=-(k_1-k_1')^2\,,$$

$T(s,t)$ is defined for $s\geqslant 4m^2$, $0\leqslant -t\leqslant s-4m^2$. In the elastic region the unitarity of the S-matrix implies

$$\operatorname{Im} T(k_1k_2k_1'k_2') = \frac{1}{2}\int Dl_1\,Dl_2\,T(k_1k_2|l_1l_2)\,T^*(l_1l_2|k_1'k_2')\,, \qquad 4m^2\leqslant s<16m^2, \tag{1.3}$$
$$Dl=\mathrm{d}l\,\theta(l)\,\delta(l^2+m^2)\,.$$

As conjectured by Mandelstam the scattering amplitude is the boundary value

$$T(s,t)=\lim_{\varepsilon\to+0}\Phi(s+i\varepsilon,t)\,, \qquad s\geqslant 4m^2,\ 0\leqslant -t\leqslant s-4m^2$$

of an analytic function of both variables s and t, regular everywhere except for the cuts

$$s\geqslant 4m^2\,,\qquad t\geqslant 4m^2\,,\qquad u=4m^2-s-t\geqslant 4m^2\,.$$

These analytic properties have been proved in some order of perturbation theory for the A^4-interaction (10).

(6) S. MacDowell: *Phys. Rev.*, **116**, 774 (1960).

(7) J. Bjorken: *Phys. Rev. Lett.*, **4**, 473 (1960); M. Nauenberg: *Thesis* (Cornell University, 1960).

(8) H. Joos: *Thesis* (Hamburg, 1961).

(9) S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).

(10) S. Mandelstam: *Phys. Rev.*, **115**, 1741 (1959) (in fourth order); G. Wanders: to be published (in sixth order).

Note added in proof. – The Mandelstam representation has been discussed in any order of perturbation theory, but the proofs given so far are incomplete. T. T. Wu

Furthermore the scattering amplitude satisfies the relations of crossing symmetry

$$\Phi(s, t) = \Phi(4m^2 - s - t, t) , \tag{1.4}$$

$$\Phi(s, t) = \Phi(s, 4m^2 - s - t) . \tag{1.5}$$

Full symmetry in the variables s, t and u follows from (4), (5), for instance

$$\Phi(s, t) = \Phi(4m^2 - s - t, t) = \Phi(4m^2 - s - t, s) = \Phi(t, s) .$$

The $t - u$ symmetry (5) implies that

$$\Phi(s; \cos\theta) = \Phi(s, t) , \qquad \cos\theta = 1 + \frac{2t}{s - 4m^2} \cdot$$

is an even function of $\cos\theta$

$$\Phi(s; \cos\theta) = \Phi(s; -\cos\theta) . \tag{1.6}$$

2. – Partial wave amplitudes.

We start by investigating the singularity of the partial wave amplitudes

$$T_l(s) = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}\cos\theta\, T(s; \cos\theta)\, P_l(\cos\theta) ,$$

at $s = 4m^2$. As a consequence of the Mandelstam representation $T_l(s)$ is the boundary value of an analytic function $\Phi_l(s)$ which is regular in the complex s-plane except for the cuts $s \leqslant 0$, $s \geqslant 4m^2$. Along the real axis we have

$$\big(\lim_{\varepsilon \to +0} \Phi_l(s - i\varepsilon)\big)^* = \lim_{\varepsilon \to +0} \Phi_l(s + i\varepsilon) ,$$

i.e. the discontinuity across the cuts is given by the imaginary part of Φ_l. The unitarity condition in the elastic region is

$$\operatorname{Im} T_l(s) = \frac{\pi}{4}\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\,|T_l(s)|^2 , \qquad 4m^2 \leqslant s < 16m^2 . \tag{2.1}$$

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and C. N. YANG: private communication; H. ENZ and J. LASCOUX: private communication; R. J. EDEN, P. V. LANDSHOFF, J. G. POLKINGHORNE and J. C. TAYLOR: preprint (1961).

Several authors (11) have remarked that the partial wave amplitudes can be continued across the branch cut $4m^2 \leqslant s < 16m^2$ into the second sheet. We will use here a different method which is analogous to Symanzik's work and leads immediately to the complete analytic behavior near $s = 4m^2$. We first define the two-particle irreducible amplitude $\Phi_l^{\text{irr}}(s)$ by the equation

$$\Phi_l^{\text{irr}}(s) = \frac{\Phi_l(s)}{1 + i\dfrac{\pi}{4}\dfrac{\sqrt{s - 4m^2}}{\sqrt{s}}\,\Phi_l(s)}\,. \tag{2.2}$$

Here $\sqrt{s - 4m^2}/\sqrt{s}$ denotes that branch which is analytic in the complex s-plane with the cuts $s \leqslant 0$, $s \geqslant 4m^2$, and is obtained by continuing $\sqrt{s - 4m^2}/\sqrt{s}$ from the upper side of the right hand cut. Then $\Phi_l^{\text{irr}}(s)$ is analytic in the s-plane with the cuts $s \leqslant 0$, $s \geqslant 4m^2$ except for poles at the zeros of the denominator (13). We further have

$$\big(\lim_{\varepsilon \to +0} \Phi_l^{\text{irr}}(s - i\varepsilon)\big)^* = \lim_{\varepsilon \to +0} \Phi_l^{\text{irr}}(s + i\varepsilon)\,.$$

In the elastic region the imaginary part vanishes identically as consequence of the unitarity condition.

$$\operatorname{Im} \Phi_l^{\text{irr}}(s) = \frac{\operatorname{Im} T_l(s) - \dfrac{\pi}{4}\dfrac{\sqrt{s - 4m^2}}{\sqrt{s}}\,|T_l(s)|^2}{\left|1 + i\dfrac{\pi}{4}\dfrac{\sqrt{s - 4m^2}}{\sqrt{s}}\,T_l(s)\right|^2} = 0\,, \qquad \text{if } 4m^2 \leqslant s < 16m^2, \tag{2.3}$$

Hence the right hand cut starts first at $s = 16m^2$ and $\Phi_l^{\text{irr}}(s)$ is regular at $s = 4m^2$ (13). On the real axis between $4m^2$ and $16m^2$ the singularities of $\Phi_l^{\text{irr}}(s)$ can only be poles. For the reciprocal

$$\Phi_l^{\text{irr}}(s)^{-1} = \frac{1 + i\dfrac{4}{\pi}\dfrac{\sqrt{s - 4m^2}}{\sqrt{s}}\,\Phi_l(s)}{\Phi_l(s)}\,, \qquad 4m^2 < s < 16m^2,$$

---

(11) M. Lévy: *Nuovo Cimento*, **15**, 13 (1959); J. Gunson and J. G. Taylor: *Phys. Rev.*, **119**, 1121 (1960); H. Beckmann and H. J. Borchers: private communications.

(12) It can be seen that this is analogous to Symanzik's definition of the two particle irreducible Green functions. In our definition, however, the particles of the intermediate state are on the mass shell.

(13) Excluding the case of a zero energy resonance we will assume that

$$1 + i\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\,T_l(s) \neq 0 \qquad \text{for } s = 4m^2.$$

is regular and vanishes at any zero s_0 of the numerator because

$$T_l(s_0) = i\,\frac{4}{\pi}\,\frac{\sqrt{s_0}}{\sqrt{s_0-4m^2}} \neq 0\,.$$

The analytic behavior of the scattering amplitude can now be obtained by solving (2) with respect to Φ_l:

$$\Phi_l(s) = \frac{\Phi_l^{\text{irr}}(s)}{1 - i\,\dfrac{\pi}{4}\,\dfrac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l^{\text{irr}}(s)}\,. \tag{2.4}$$

This yields

$$\Phi_l(s) = F_l(s) + i\,\frac{\pi}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G_l(s)\,, \tag{2.5}$$

where

$$F_l(s) = \frac{\Phi_l^{\text{irr}}(s)}{1+\pi^2\,\dfrac{s-4m^2}{16s}\,\Phi_l^{\text{irr}}(s)^2} \tag{2.6}$$

$$G_l(s) = \frac{\Phi_l^{\text{irr}}(s)^2}{1+\pi^2\,\dfrac{s-4m^2}{16s}\,\Phi_l^{\text{irr}}(s)^2}\,, \tag{2.7}$$

are again regular in the complex s-plane with the cuts $s\leqslant 0$, $s\geqslant 16m^2$ except for poles.

As F and G are regular at $s=4m^2$ relation (5) gives explicitly the singularity at $s=4m^2$. Φ_l has a twofold branch point as $s=4m^2$ and can be continued everywhere on the Riemann surface of $\sqrt{s-4m^2}$ with the cuts $s\leqslant 0$, $s\geqslant 16m^2$ except for poles [14].

Let us distinguish the values of $\Phi_l(s)$ on the second sheet from the values on the first sheet by an index 2

$$\Phi_l(s) = F_l(s) + i\,\frac{\pi}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G_l(s)\,, \tag{2.8}$$

$$\Phi_l^{(2)}(s) = F_l(s) - i\,\frac{\pi}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G_l(s)\,, \tag{2.9}$$

with $\sqrt{s-4m^2}/\sqrt{s}$ defined as in eq. (2). According to (3), (6) and (7) the functions F_l and G_l are real in the elastic region. Hence taking the limit

$$\lim_{\varepsilon\to+0}\Phi_l(s+i\varepsilon) = T_l(s)$$

[14] Using different methods this result has been found independently in ref. [1-3].

of eq. (8) we obtain

$$\text{(2.10)} \qquad \operatorname{Re} T_l(s) = F_l(s)\,, \qquad \text{for } 4m^2 \leqslant s < 16m^2\,,$$

$$\text{(2.11)} \qquad \operatorname{Im} T_l(s) = \frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G_l(s)\,.$$

F_l and G_l can therefore be defined by the analytic continuations of the dispersive and the absorptive part from the elastic region of the real axis into the complex s-plane. The dispersive part is regular at $s = 4m^2$, the absorptive part is singular at zero energy containing $\sqrt{s-4m^2}$ as a factor.

Applying the limit $\operatorname{Im} s \to +0$ to (9) we get

$$\text{(2.12)} \qquad \lim_{\varepsilon\to+0} \Phi_l^{(2)}(s+i) = F_l(s) - i\,\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G_l(s) = T_l^*(s)\,,$$

$$\text{for } 4m^2 \leqslant s < 16m^2\,.$$

Using (11) and (12) we obtain by continuing the unitarity condition (1)

$$\text{(2.13)} \qquad \Phi_l(s) = \Phi_l^{(2)}(s) = i\,\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G_l(s) = i\,\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l(s)\,\Phi_l^{(2)}(s)\,,$$

and [15]

$$\text{(2.14)} \qquad \Phi_l^{(2)}(s) = \frac{\Phi_l(s)}{1 + i\,\dfrac{\pi}{2}\dfrac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l(s)}\,.$$

This relation has been used by GUNSON and TAYLOR [11] for constructing the analytic continuation of the scattering amplitude across the two particle branch cut.

The possibility of poles for $\Phi_l^{(2)}(s)$ has been well discussed by BLANKENBECLER, GOLDBERGER, MACDOWELL and TREIMAN (1). We only add some remarks concerning the equations (4), (8) and (9). Poles of F and G must compensate in (8) as $\Phi_l(s)$ cannot have a pole on the first sheet. For $4m^2 \leqslant s < 16m^2$ $\Phi_l(s)$ cannot have a pole either because $\Phi_l^{\text{irr}}(s)$ is real in the denominator of (4). On the second sheet, however, poles occur at the zeros of

$$1 - i\,\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l(s) \qquad \text{or} \qquad 1 - i\,\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l^{\text{irr}}(s)\,.$$

---

[15] Note that the eq. (4) and (14) differ only by a factor 2 in the denominator.

(It can easily be seen that both expressions must have the same zeros). As well known [16] these poles indicate resonance states of the system.

The two-particle irreducible amplitude has a simple physical meaning. It coincides with the corresponding K-matrix element in the elastic region

$$\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l^{\text{irr}}(s) = K_l(s) = 2\,\operatorname{tg}\delta_l(s)\,, \qquad \text{for } 4m^2 \leqslant s < 16m^2, \tag{2.15}$$

where

$$K_l = 2i\,\frac{1-S_l}{1+S_l}\,, \qquad S_l = \exp\left[2i\delta_l\right] = 1 + i\,\frac{\pi}{2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,T_l\,.$$

Φ_l^{irr} can be defined by the analytic continuation of $(4/\pi)\big(\sqrt{s}/\sqrt{s-4m^2}\big)\,K_l$ from the elastic region into the complex s-plane. Above $s=16m^2$ the amplitude $K_l(s)$ and

$$\lim_{\varepsilon\to+0}\frac{\pi}{2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l^{\text{irr}}(s+i\varepsilon)\,,$$

are different. In the elastic region of the real axis $\Phi_l^{\text{irr}}(s)$ has a pole for $\delta_l(s)=\pi/2$. As $\Phi_l^{\text{irr}}(s)$ is regular at $s=4m^2$ it follows that

$$\operatorname{tg}\delta(s) = \sqrt{s-4m^2}\,\big(c_0 + c_1(s-4m^2) + \ldots\big)\,,$$

converges in the neighborhood of $s=4m^2$ [13].

3. – Partial wave expansion.

As consequence of the Mandelstam representation the partial wave expansion

$$\Phi(s;\,\cos\theta) = \sum\,(2l+1)\,\Phi_l(s)\,P_l(\cos\theta) \tag{3.1}$$

converges inside the ellipse $E_1(s)$ of the $\cos\theta$-plane with foci at ± 1 and semi-major axis

$$a_1(s) = \sqrt{1 + 8m^2\,\frac{|s| + \operatorname{Re}s}{|s-4m^2|^2}}\,, \tag{3.2}$$

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[16] R. E. PEIERLS: *Proc. of the Glasgow Conference* (1954), p. 296; J. GUNSON and J. C. TAYLOR: *Phys. Rev.*, **119**, 1121 (1960).

for any value of s except $s \leqslant 0$. On the boundary of $E_1(s)$ lie the starting points

$$\cos\theta = \pm\left(1 + \frac{8m^2}{s-4m^2}\right),$$

of the cuts

(3.3) $$\cos\theta = \pm\left(1 + \frac{2t}{s-4m^2}\right), \qquad t \geqslant 4m^2.$$

We now define Φ^{irr}, F and G as functions of s and $\cos\theta$ by the partial wave expansions

(3.4) $$\Phi^{\text{irr}}(s;\cos\theta) = \sum (2l+1)\,\Phi_l^{\text{irr}}(s)\,P_l(\cos\theta)\,,$$

(3.5) $$F(s;\cos\theta) = \sum (2l+1)\,F_l(s)\quad P_l(\cos\theta)\,,$$

(3.6) $$G(s;\cos\theta) = \sum (2l+1)\,G_l(s)\quad P_l(\cos\theta)\,.$$

We first determine the convergence domain of (4) ([17]). If s is not negative or zero and if none of the $\Phi_l^{\text{irr}}(s)$ has a pole the expansion (4) converges inside the ellipse of the $\cos\theta$-plane with foci at ± 1 and semi axes a, b given by

$$(a+b)^{-1} = \overline{\lim_{l\to\infty}}\,|\Phi_l^{\text{irr}}(s)|^{1/l} = \overline{\lim_{l\to\infty}}\,\frac{|\Phi_l(s)|^{1/l}}{\left|1 + i\,\dfrac{\pi}{4}\,\dfrac{\sqrt{s-4m^2}}{\sqrt{s}}\,\Phi_l(s)\right|^{1/l}} = \overline{\lim_{l\to\infty}}\,|\Phi_l(s)|^{1/l}$$

$$\left(\text{because } \lim_{l\to\infty}\Phi_l(s) = 0\right).$$

Similarly

$$\overline{\lim_{l\to\infty}}\,|F_l(s)|^{1/l} = \overline{\lim_{l\to\infty}}\,|\Phi_l^{\text{irr}}(s)|^{1/l}\,.$$

Hence the expansions (1), (4) and (5) have the same convergence domain $E_1(s)$ in the $\cos\theta$-plane. For the expansion (6) we get convergence in a larger ellipse $E_2(s)$ with semi-major axis

(3.7) $$a_2(s) = 2a_1^2(s) - 1 = 1 + 16m^2\,\frac{|s| + \operatorname{Re} s}{|s-4m^2|^2}\,,$$

because

(3.8) $$\overline{\lim_{l\to\infty}}\,|G_l(s)|^{1/l} = \left(\overline{\lim_{l\to\infty}}\,|\Phi_l^{\text{irr}}(s)|^{1/l}\right)^2 = \left(\overline{\lim_{l\to\infty}}\,|\Phi_l(s)|^{1/l}\right)^2,$$

([17]) The following method has been used in ref. ([2]) to define the scattering amplitude on the second sheet directly by the Legendre expansion with coefficients (2.14).

If one or more partial wave amplitudes $\Phi^{\text{irr}}_{l_1}(s), \ldots, \Phi^{\text{irr}}_{l_r}(s)$ have a pole at $s=a$ with residuum c_α and order n_α we define Φ'_{irr} by

$$\Phi_{\text{irr}}(s;\cos\theta) = \Phi'_{\text{irr}}(s;\cos\theta) + \sum_\alpha \frac{c_\alpha P_{l_\alpha}(\cos\theta)}{(s-a)^{n_\alpha}}\,. \tag{3.9}$$

The partial wave expansion of Φ'_{irr} converges at $s=a$ in the convergence domain $E_1(a)$ of (1). In the same way the poles of F and G can be treated.

The poles of Φ_{irr} (and correspondingly the poles of F and G) never cumulate in the s-plane except for $s=\infty$ or on the cuts $s<0$, $s\geqslant 16m^2$. This can be shown in the following way. The partial wave amplitudes converge to zero

$$\lim_{l\to\infty} T_l(s) = 0 \tag{3.10}$$

uniformly in any closed and finite region C of the s-plane which contains no point $s\leqslant 0$ [18]. As $|i(4/\pi)(\sqrt{s}/\sqrt{s-4m^2})|$ has a positive lower bound in C, only a finite number of $\Phi^{\text{irr}}_l(s)$ can have a pole in C. Therefore the cumulation points of the poles must lie on the cuts or at infinity. In particular any infinite sequence of poles with increasing values of l must approach $s=\infty$ or points of the negative real axis $s\leqslant 0$ for $l\to\infty$.

We thus arrive at the result that the expansions (4) (5) and (6) define Φ_{irr}, F and G as analytic functions of s and $\cos\theta$ regular in the convergence domain of (1) except for the cuts

$$s \leqslant 0\,, \qquad s \geqslant 16m^2$$

and poles of the form

$$\sum_\alpha \frac{c_\alpha P_{l_\alpha}(\cos\theta)}{(s-a)^{n_\alpha}}\,. \tag{3.11}$$

Φ_{irr}, F and G are real for $0<s<16m^2$ and $-1\leqslant\cos\theta\leqslant+1$.

By inserting (2.8) into (3.1) we get

$$\Phi(s;\cos\theta) = F(s;\cos\theta) + i\,\frac{\pi}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G(s;\cos\theta)\,. \tag{3.12}$$

[18] The uniform convergence of (10) follows from the analytic properties of $\Phi(s;\cos\theta)$. If the closed domain C is bounded and contains no points $s\leqslant 0$ a fixed ellipse E with foci ± 1 can be chosen such that $\Phi(s;\cos\theta)$ is regular in both variables inside and on the boundary of C and E. The expansion (1) converges uniformly in C and E, hence (10) holds uniformly in C. The author is grateful to Dr. Jost for pointing out this to him.

This relation gives the complete analytic behavior of the scattering amplitude at $a=4m^2$ in both variables s and $\cos\theta$. On the second sheet

$$\Phi^{(2)}(s;\cos\theta)=F(s;\cos\theta)-i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G(s;\cos\theta)= \tag{3.13}$$

$$=\Phi(s;\cos\theta)-i\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G(s;\cos\theta),$$

is defined in the convergence domain of (6). Hence $\Phi^{(2)}(s;\cos\theta)$ is regular in s and $\cos\theta$ if

i) s is arbitrary except for the cuts $s\leqslant 0$, $s\geqslant 4m^2$ and poles of the form (9),

ii) $\cos\theta$ is inside the ellipse $E_2(s)$ except for the cuts

$$\cos\theta=\pm\left(1+\frac{2t}{s-4m^2}\right),\qquad t\geqslant 4m^2.$$

If $\cos\theta$ is kept fixed and $\Phi^{(2)}(s;\cos\theta)$ is considered as function of s alone, condition ii) restricts the regularity domain of the complex s-plane. But if $\cos\theta$ is real and between -1 and $+1$ the function $\Phi^{(2)}(s;\cos\theta)$ is regular in the whole complex s-plane except for the cuts $s\leqslant 0$, $s\geqslant 4m^2$ and poles.

In the elastic region we have

$$\operatorname{Re}T(s;\cos\theta)=F(s;\cos\theta),\qquad 4m^2\leqslant s<16m^2,\ -1\leqslant\cos\theta\leqslant+1, \tag{3.14}$$

$$\operatorname{Im}T(s;\cos\theta)=\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G(s;\cos\theta), \tag{3.15}$$

The poles of F and $\Phi^{(2)}$ are uniquely determined by the poles of G. Let $G_{l_1}(s), \ldots, G_{l_2}(s)$ denote all partial wave amplitudes which have a pole at $s=a$, then $G(s;\cos\theta)$ takes the form

$$G(s;\cos\theta)=\sum_{\alpha}\frac{c_\alpha P_{l_\alpha}(\cos\theta)}{(s-a)^{n_\alpha}}+G'(s;\cos\theta), \tag{3.16}$$

with G' regular at $s=a$. Using the fact that the scattering amplitude has

no poles on the first sheet we obtain for F and $\Phi^{(2)}$

$$F(s;\cos\theta)=\Phi-i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G=-i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\sum_\alpha\frac{c_\alpha P_{l_\alpha}(\cos\theta)}{(s-a)^{n_\alpha}}+F'(s;\cos\theta)$$

$$\Phi^{(2)}(s;\cos\theta)=\Phi-i\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G=$$
$$=-i\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\sum_\alpha\frac{c_\alpha P_{l_\alpha}(\cos\theta)}{(s-a)^{n_\alpha}}+\Phi^{(2)\prime}(s;\cos\theta)\,,$$

F' and $\Phi^{(2)\prime}$ are regular at $s=a$. The coefficients of the principal parts at $s=a$ are linear combinations of the P_l involving the constants c_α, $\sqrt{a-4m^2}/\sqrt{a}$ and derivatives of $\sqrt{s-4m^2}/\sqrt{s}$ at $s=a$.

Outside the domain E_2 of the $\cos\theta$-plane the partial wave expansion of G diverges and cannot be used for continuing $\Phi^{(2)}$. We therefore return to the integral form (1.3) of the unitarity condition which will be discussed in the following section.

4. – Analytic continuation beyond the convergence domain of partial wave expansion.

In the elastic region the unitarity condition (1.3) can be written as

$$\operatorname{Im}T(s;\cos\theta)=$$
$$-\frac{1}{8}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\int_{-1}^{+1}\mathrm{d}\cos\varphi\int_{-1}^{+1}\mathrm{d}\cos\psi\,\frac{T(s;\cos\varphi)\,T^*(s;\cos\psi)}{\sqrt{-k(\cos\varphi,\cos\psi,\cos\theta)}}\,\theta(-k(\cos\varphi,\cos\psi,\cos\theta))\,,$$

$$k(\xi,\eta,\zeta)=\xi^2+\eta^2+\zeta^2-2\xi\eta\zeta-1\,,\qquad -1\leqslant\cos\theta\leqslant+1\,,\tag{4.1}$$

if $4m^2\leqslant s<16m^2$.

For fixed $\cos\theta$ both sides of (1) are boundary values of analytic functions in s because

$$T(s;\cos\theta)=\lim_{\varepsilon\to+0}\Phi(s+i\varepsilon;\cos\theta)\qquad\text{for } s\geqslant4m^2,$$

$$T^*(s;\cos\theta)=\lim_{\varepsilon\to+0}\Phi^{(2)}(s+i\varepsilon;\cos\theta)\quad\text{for } 4m^2\leqslant s<16m^2,\ -1\leqslant\cos\theta\leqslant+1.$$

As we have seen in the preceding section the function $\Phi^{(2)}(s;\cos\theta)$ is regular in the complex s-plane except for the cuts $s\leqslant0$, $s\geqslant4m^2$ and the poles of $\Phi^{(2)}_l(s)$ if $\cos\theta$ is held fixed and $-1\leqslant\cos\theta\leqslant+1$. Equation (1) can therefore

be continued to complex s-values by

$$(4.2)\qquad i\,\frac{\pi}{2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G(s;\cos\theta)=\Phi(s;\cos\theta)-\Phi^{(2)}(s;\cos\theta)=$$

$$=\frac{i}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\int\limits_{-1}^{+1}\mathrm{d}\cos\varphi\int\limits_{-1}^{+1}\mathrm{d}\cos\psi\,\frac{\Phi(s;\cos\varphi)\,\Phi^{(2)}(s;\cos\psi)}{\sqrt{-k(\cos\varphi,\cos\psi,\cos\theta)}}\,\theta(-k(\cos\varphi,\cos\psi,\cos\theta))\,.$$

$$-1\leqslant\cos\theta\leqslant+1\,.$$

If Φ is considered as given this is a linear integral equation for $\Phi^{(2)}$ which can be used for continuing $\Phi^{(2)}$ in $\cos\theta$.

We already know that $\Phi^{(2)}(s;\cos\theta)$ is analytic in $\cos\theta$ except for $s\leqslant 0$ and poles in s. The regularity domain in the $\cos\theta$-plane we obtained in the preceding section was the ellipse $E_2(s)$ with foci ± 1, semi-major axis $2a_1^2(s)-1$ (eq. (3.7)) and the cuts

$$(4.3)\qquad \cos\theta=\pm\left(1+\frac{2t}{s-4m^2}\right),\qquad t\geqslant 4m^2\,.$$

It will be convenient to introduce the complex scattering angle $\alpha=i\theta$. In this variable the cut (3) is given by

$$(4.4)\qquad \alpha(s,t)=\cosh^{-1}\left(1+\frac{2t}{s-4m^2}\right),\qquad t\geqslant 4m^2\,,$$

with starting point at

$$(4.5)\qquad \alpha(s)=\alpha(s,4m^2)=\cosh^{-1}\left(1+\frac{8m^2}{s-4m^2}\right).$$

The ellipses E_1, E_2 have the semi-major axis (eq. (3.2) and (3.7)):

$$a_1(s)=\cosh\operatorname{Re}\alpha(s)\,,\qquad a_2(s)=\cosh 2\operatorname{Re}\alpha(s)\,.$$

The ellipse with foci ± 1 and semi-major axis

$$(4.6)\qquad a_n(s)=\cosh n\operatorname{Re}\alpha(s)$$

will be denoted as $E_n(s)$ throughout the work that follows.

4·1. *Continuation into the ellipse E_3 with respect to* $\cos\theta$. – Let us keep s at a fixed point of the complex s-plane $s\leqslant 0$ and poles of $\Phi_l^{(2)}(s)$. We include the case that s lies on the upper or lower side of the cut $s\geqslant 4m^2$. We will now use eq. (2) to continue $\Phi^{(2)}(s;\cos\theta)$ analytically in $\cos\theta$ from the ellipse $E_2(s)$ to the larger ellipse $E_3(s)$ by generalizing a method of Mandelstam [19].

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(19) S. MANDELSTAM: *Nuovo Cimento*, **15**, 658 (1960).

We first express Φ and $\Phi^{(2)}$ by the Cauchy integrals

$$
(4.7)\qquad \left|\begin{aligned}
\Phi\ (s;\cos\theta) &= \frac{1}{2\pi i}\oint_C \frac{\Phi(s;\zeta)}{\cos\theta-\zeta}\,\mathrm{d}\zeta\,,\\
\Phi^{(2)}(s;\cos\theta) &= \frac{1}{2\pi i}\oint_C \frac{\Phi^{(2)}(s;\zeta)}{\cos\theta-\zeta}\,\mathrm{d}\zeta\,,
\end{aligned}\right.
$$

along the boundary C of E_2 excluding the cuts (3). Inserting (7) into the right hand side of (2) we obtain

$$
(4.8)\qquad i\,\frac{\pi}{2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G(s;\cos\theta)=\Phi(s;\cos\theta)-\Phi^{(2)}(s;\cos\theta)=
$$

$$
=-\frac{i}{16\pi^2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\oint_C \mathrm{d}\xi\oint_C \mathrm{d}\eta\,\Phi(s;\xi)\,\Phi^{(2)}(s;\eta)\,H(\xi,\eta,\cos\theta)\,,
$$

where the function H is defined by

$$
(4.9)\qquad H(\xi,\eta,z)=\int_{-1}^{+1}\frac{\mathrm{d}\xi'}{\xi'-\xi}\int_{-1}^{+1}\frac{\mathrm{d}\eta'}{\eta'-\eta}\,\frac{\theta(-k(\xi',\eta',z))}{|\sqrt{-k(\xi',\eta',z)}|}\,.
$$

for $-1\leqslant z\leqslant+1$ and ξ,η off the cuts $-1\leqslant\xi\leqslant+1$, $-1\leqslant\eta\leqslant+1$.

It will be proved in this section that (8) defines an analytic continuation of $G(s;\cos\theta)$ into the ellipse E_3 (except for a certain domain where $\Phi^{(2)}$ may have singularities).

The analytic properties of the function H in the variable z have been given by MANDELSTAM [20]. H satisfies a dispersion relation in z

$$
(4.10)\qquad H(\xi,\eta,z)=-2\pi\int_{\xi\eta+\sqrt{\xi^2-1}\sqrt{\eta^2-1}}^{\infty}\frac{\mathrm{d}\zeta}{\zeta-z}\,\frac{1}{\sqrt{k(\xi,\eta,z)}}\,,
$$

and is given explicitly by

$$
H(\xi,\eta,z)=-\frac{\pi}{\sqrt{k(\xi,\eta,z)}}\ln\frac{z-\xi\eta+\sqrt{k(\xi,\eta,z)}}{z-\xi\eta-\sqrt{k(\xi,\eta,z)}}\,.
$$

In (10) the square roots $\sqrt{\xi^2-1}$, $\sqrt{\eta^2-1}$ denote those branches which are positive for $\xi>1$, $\eta>1$ and regular in the complex plane except for a cut

([20]) S. MANDELSTAM: *Phys. Rev.*, **115**, 1741 (1959).

from -1 to $+1$ along the real axis. $H(\xi, \eta, z)$ is regular in the complex z-plane except for a singularity at

$$z = \xi\eta + \sqrt{\xi^2 - 1}\sqrt{\eta^2 - 1} = \cosh(\alpha + \beta)\,, \qquad (\xi = \cosh\alpha,\ \eta = \cosh\beta), \tag{4.11}$$

with a cut leading from (11) to infinity. The cut as well as the integration in (10) may be taken along any path outside the ellipse with foci ± 1 and boundary point $\xi\eta + \sqrt{\xi^2-1}\sqrt{\eta^2-1}$.

The only $\cos\theta$-dependent term in the integrand of (8) is $H(\xi, \eta, \cos\theta)$. Here ξ and η lie on the integration path C, *i.e.* on the boundary of the ellipse E_2 and (inside E_2) on both sides of the cuts

$$\begin{cases} \xi = \pm\left(1 + \dfrac{2t_1}{s - 4m^2}\right) = \pm\cosh\alpha(s, t_1)\,, & t_1 \geqslant 4m^2, \\ \eta = \pm\left(1 + \dfrac{2t_2}{s - 4m^2}\right) = \pm\cosh\alpha(s, t_2)\,. & t_2 \geqslant 4m^2. \end{cases} \tag{4.12}$$

Let D_2 denote the domain of all points

$$D_2^{\pm}: \qquad \pm\cosh\big(\alpha(s, t_1) + \alpha(s, t_2)\big)\,, \qquad t_1, t_2 \geqslant 4m^2. \tag{4.13}$$

Then the singularity (11) of $H(\xi, \eta, \cos\theta)$ lies (ξ, η both on C) either

i) outside or on the boundary of the ellipse E_3 with foci ± 1 and semi-major axis $\cosh\big(3a(s)\big)$

or

ii) inside the ellipse E_3 *and* inside the domain D_2^+ or D_2^-.

We now define the analytic continuation of the integral (8) as follows. For any given parameter ξ and η we continue H from $-1 \leqslant \cos\theta \leqslant +1$ into the complex $\cos\theta$-plane choosing a cut from

$$\xi\eta + \sqrt{\xi^2 - 1}\sqrt{\eta^2 - 1}\,,$$

to infinity which lies outside E_3 in case i) and inside D_2^- or D_2^- in case ii).

Then (8) defines an analytic continuation of G in $\cos\theta$ from E_2 into the domain $E_3 - D_2^+ - D_2^-$. Hence for given s (except $s \leqslant 0$ and poles) the function $G(s; \cos\theta)$ is analytic in $\cos\theta$ in the larger ellipse $E_3(s)$ except for the domains $D_2^+(s)$ and $D_2^-(s)$.

The (in general) two-dimensional region D_2^+ lies outside E_2 and comes to an apex at the boundary point

$$\cos\theta = \cosh\big(2\alpha(s)\big)\,. \tag{4.14}$$

For $s > 4m^2$ the regions D_2 reduce to normal cuts along the real axis starting at $\pm(2a_1^2(s)-1)$.

4'2. *Analyticity in E_3 with respect to both variables.* – Within the domain $E_3 - D_2^+ - D_2^-$ we also have analyticity in s. In order to show this we shift the path C in eq. (8) slightly inwards to a new integration path C' which lies completely inside the regularity domain E_2 and does not touch the cuts (12). For any interior point $\cos\theta = z_0$ of $E_3 - D_2^+ - D_2^-$ the curve C' can be chosen close enough to C for the integrand of (8) still to be regular in $\cos\theta$ at z_0. We now consider an arbitrary point s_0 of the s-plane (except the cuts $s < 0$, $s \geq 4m^2$ and poles) and keep $\cos\theta$ fixed at an interior point z_0 of $E_3(s_0) - D_2^+(s_0) - D_2^-(s_0)$. Then the integrand of (8) is regular at s_0 for any parameter value ξ, η on C'. $G(s; \cos\theta)$ is hence regular at $s = s_0$ for $\cos\theta = z_0$.

We thus arrive at the result that the scattering amplitude

$$\Phi^{(2)}(s; \cos\theta) = \Phi(s; \cos\theta) - i\,\frac{\pi}{2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,G(s; \cos\theta)\,, \tag{4.15}$$

is regular on the second sheet in both variables s and $\cos\theta$ if

i) s is not a pole of the partial wave amplitudes $\Phi_l^{(2)}(s)$ and does not lie on the cuts $s \leq 0$ and $s \geq 4m^2$.

ii) $\cos\theta$ inside $E_3(s)$ except for the regions $D_2(s)$ and the normal cuts

$$\cos\theta = \pm\left(1 + \frac{2t}{s-4m^2}\right), \qquad t \geq 4m^2;$$

The boundary values of $\Phi^{(2)}$ along the cut $s > 4m^2$ still have the analytic properties ii) in $\cos\theta$. For $F = \frac{1}{2}(\Phi + \Phi^{(2)})$ the same analytic properties i) and ii) follow.

We must still show that $G(s; \cos\theta)$ and $F(s; \cos\theta)$ are regular in s along $4m^2 \leq s < 16m^2$ if $\cos\theta$ belongs to the new domain $E_3 - D_2^+ - D_2^-$. As both sides of

$$\lim_{\varepsilon\to+0} G(s+i\varepsilon; \cos\theta) = \lim_{\varepsilon\to+0} G(s-i\varepsilon; \cos\theta) \qquad 4m^2 \leq s < 16m^2 \tag{4.16}$$

are analytic functions of $\cos\theta$ eq. (16) can be continued from E_2 into $E_3 - D_2^+ - D_2^-$. Let s_0 be a real value between $4m^2$ and $16m^2$ and $\cos\theta = z_0$ be an interior point of $E_3(s_0) - D_2^+(s_0) - D_2^-(s_0)$. Then z_0 lies also in $E_3(s) - D_2^+(s) - D_2^-(s)$ for any s belonging to a finite strip along $4m^2 \leq s < s_0$: $G(s; z_0)$ is a regular function of s in the cut strip and satisfies (16) along the cut. Therefore G (and correspondingly F) is regular in s along $4m^2 \leq s < 16m^2$ for all $\cos\theta$ belonging to $E_3(s) - D_2^+(s) - D_2^-(s)$.

Finally we discuss the continuation of the pole term (3.16) into $E_3 - D_2^+ - D_2^-$. Multiplying both sides of (2) by $(s-a)^n$ (n is the maximum order of the poles at $s=a$) the integrand becomes regular at $s=a$. We obtain regularity for

$$\hat{G}'(s;\cos\theta) = (s-a)^n G(s;\cos\theta)$$

at $s=a$ and $\cos\theta = z_0$ inside $E_3(a) - D_2^+(a) - D_2^-(a)$. The principal part of G at $s=a$ is determined by G' and the first $n-1$ derivatives at $s=a$. As these are analytic functions in $\cos\theta$ eq. (3.16) holds for any $\cos\theta$ inside the new regularity domain.

4·3. *Continuation beyond the ellipse* E_3. – Analytic continuation of $\Phi^{(2)}$ into the ellipse E_4 is obtained by applying the same method once more. Φ and $\Phi^{(2)}$ are first expressed by Cauchy integrals along the boundary of their regularity domain in E_3, the result is inserted into the right-hand side of (2), etc. This procedure can be repeated arbitrarily often and leads finally to the following result.

$\Phi^{(2)}(s;\cos\theta)$ is analytic in s and $\cos\theta$ except for

(a) the normal cuts

(a_1) $\qquad s \geqslant 4m^2,$

(a_2) $\qquad \cos\theta = \pm\left(1 + \dfrac{2t}{s-4m^2}\right).$

(Along the cuts (a_1) the boundary values of $\Phi^{(2)}$ are still analytic in $\cos\theta$ except for the cuts (a_2));

(b) the kinematical cut $\quad s \leqslant 0,$

(c) poles of the form $\displaystyle\sum_\alpha \frac{c_\alpha P_{l_\alpha}(\cos\theta)}{(s-a)^{n_\alpha}}$

(d) the domains $D_n^\pm$ defined by

$$D_n^\pm:\quad \cos\theta = \pm\cosh\big(\alpha(s,t_1) + \ldots + \alpha(s,t_n)\big),\quad t_i \geqslant 4m^2,\quad n = 1, 2, 3, \ldots . \tag{4.17}$$

For $F(s;\cos\theta)$ and $G(s;\cos\theta)$ the same analytic properties follow except that the cut (a_1) starts at $16m^2$ and that G does not have the normal t- and u-cut (a_2).

5. – Landau singularities.

In the preceding section we have seen that the scattering amplitude $\Phi(s;\cos\theta)$ can be continued everywhere on the second sheet except for certain domains $D_n(s)$ of the $\cos\theta$-plane. These domains are in general two dimensional, but reduce to normal cuts for $s>0$. The domain $D_n(s)$ lies outside the ellipse $E_n(s)$ and comes to an apex at the boundary point

$$\cos\theta = \pm\cosh\big(n\,\alpha(s)\big) \quad \text{of} \quad E_n(s)\,, \tag{5.1}$$

where $\Phi^{(2)}(s;\cos\theta)$ has a branch point. These singularities have been found in perturbation theory by OKUN-RUDIK and independently by ENZ-LASCOUX [21] For $n=2$ it can be easily seen that the singularity (1) is actually present because the partial wave expansion of G diverges outside E_2 (because of (3.8)) and G is regular elsewhere on the boundary of E_2.

The singularities of $\Phi^{(2)}$ inside the domain D_n can be discussed by the following addition rule. If $\Phi(s;\xi)$ and $\Phi^{(2)}(s;\eta)$ have a singularity at

$$\xi = \cosh\alpha\,, \qquad \eta = \cosh\beta\,;$$

then in general the integral (4.8) has a singularity at

$$\cos\theta = \pm\cosh(\alpha+\beta)\,.$$

As $\Phi(s;\cos\theta)$ and $\Phi^{(2)}(s;\cos\theta)$ have singularities at

$$\cos\theta = 1 + \frac{8n^2m^2}{s-4m^2} = \cosh\alpha_n(s)\,, \qquad n=1, 2, \dots$$

$\Phi^{(2)}(s;\cos\theta)$ is expected to have additional singularities at

$$\cos\theta = \pm\cosh\big(n_{\nu_1}\alpha_{\nu_1}(s)+\dots+n_{\nu_r}\alpha_{\nu_r}(s)\big) \qquad n_{\nu_i}=1, 2, \dots \tag{5.2}$$

These singularities apparently lie within the domain $D_n^{\pm}$, they have also been found in perturbation theory [21].

This is, however, not the most general type of singularity occurring in D_n.

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[21] L. D. LANDAU: *Nucl. Phys.*, **13**, 181 (1959); L. P. OKUN and A. P. RUDIK: *Nucl. Phys.*, **14**, 261 (1960); H. ENZ and J. LASCOUX: private communication; see also ref. [1].

In order to show this let us consider a circle A in the complex θ-plane with center at $1+(8m^2/(s-4m^2))$. A is supposed to be so small that it contains no furher singularity on the cut $t \geqslant 4m^2$.

Let B denote the region of all

$$\cos\theta = \xi\eta + \sqrt{\xi^2-1}\sqrt{\eta^2-1}\,,$$

which can be formed by any points ξ, η belonging to A. Inside B the region D_2 can be reduced to normal branch cuts if both cuts

$$(5.3)\qquad \begin{cases} \xi = 1+\dfrac{2t_1}{s-4m^2} = \cosh\alpha(s,t_1) & t_1 \geqslant 4m^2\,, \\ \eta = 1+\dfrac{2t_2}{s-4m^2} = \cosh\alpha(s,t_2)\,, & t_2 \geqslant 4m^2\,. \end{cases}$$

can be shifted to

$$(5.4)\qquad \begin{cases} \xi = \cosh\big(\varrho_1 + i\,\mathrm{Im}\,\alpha(s)\big) \\ \eta = \cosh\big(\varrho_2 + i\,\mathrm{Im}\,\alpha(s)\big) \end{cases} \qquad \varrho_{1,2} \geqslant \mathrm{Re}\,\alpha(s)$$

without crossing a pole or a Landau singularity. Using (4.8) with an appropriate integration path C we obtain $\Phi^{(2)}(s;\cos\theta)$ as regular function in B except for the cut

$$\cos\theta = \cosh\big(2\varrho + 2i\,\mathrm{Im}\,\alpha(s)\big) \qquad \varrho \geqslant \mathrm{Re}\,\alpha(s).$$

If, however, poles are crossed on the second sheet of $\sqrt{t-4m^2}$ when shifting both cuts from (3) to (4) one gets additional branch points in D_2. These singularities have not been found in perturbation theory.

We remark that for $s\to 0$ the branch points (1) all approach $\cos\theta = \pm 1$. The analytic behavior of the scattering amplitude at the kinematical branch point $s=0$ (on the second sheet) therefore seems to be rather complicated.

In terms of t the Landau singularities (1) are (for fixed s)

$$(5.5)\qquad \begin{cases} n=2: & t_2^+(s) = 16m^2 + \dfrac{64m^4}{s-4m^2}\,, \\ n=3: & t_3^+(s) = 36m^2 + \dfrac{384m^4}{s-4m^2} + \dfrac{1024m^6}{(s-4m^2)^2}\,, \\ & \dots\dots\dots\dots\dots \end{cases}$$

and

$$t_n^-(s) = 4m^2 - s - t_n^+(s) .$$

In general we obtain for $t_n^+(s)$ a polynomial in $(s-4m^2)^{-1}$ with a constant term given by

$$\lim_{s\to\infty} t_n^+(s) = 4n^2m^2 .$$

(5) shows that the scattering amplitude has complex singularities on the second sheet of $\sqrt{s-4m^2}$.

The singularity curves $t_2^\pm(s)$ partly coincide with the boundary curve of the region where the spectral function of the Mandelstam representation is different from zero. This can be seen as follows: if $4m^2 < s < 16m^2$ the function $G(s, t)$ is regular in the complex s-plane except for the cuts

$$t \geqslant t_2^+(s) , \qquad t \leqslant t_2^-(s) .$$

Hence $G(s, t)$ satisfies a dispersion relation in t with a weight function vanishing for

$$t_2^-(s) < t < t_2^+(s) .$$

In the elastic region the weight function of $G(s, t)$ is (apart from the factor $(\pi/4)\sqrt{s-4m^2}/\sqrt{s}$) identical with Mandelstam's spectral function as follows from (3.12)

$$\operatorname{Im} T(s, t) = \frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, G(s, t) , \qquad 4m^2 \leqslant s < 16m^2 .$$

6. – Analytic properties of the K-matrix element in both variables.

The two-particle irreducible amplitude

$$T_{\text{irr}}(k_1k_2\,k_1'k_2') = \delta(k_1+k_2-k_1'-k_2')\,T_{\text{irr}}(s;\,\cos\theta)$$

as defined by (3.4) and (2.4)

(2.4) $$T_l(s) = T_l^{\text{irr}}(s) + i\,\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,T_l(s)\,T_l^{\text{irr}}(s) ,$$

satisfies Heitler's integral equation ([22,23])

(6.1) $$T(k_1k_2\,|\,k_1'k_2') = T_{\text{irr}}(k_1k_2\,|\,k_1'k_2') + \frac{i}{2}\int Dl_1\,Dl_2\,T(k_1k_2\,|\,l_1l_2)\,T_{\text{irr}}(l_1l_2\,|\,k_1'k_2')$$

([22]) W. Heitler: *Proc. Camb. Phil. Soc.*, **37**, 291 (1941); M. L. Goldberger: *Phys. Rev.*, **84**, 60 (1951), this paper contains further references.

([23]) In our approach Heitler's integral equation is the analogue to the Bethe-Salpeter type equations used by Symanzik in ref. ([4]).

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or

$$(6.2)\qquad T(s;\cos\theta) = T_{\text{irr}}(s;\cos\theta) +$$
$$-\frac{i}{8}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\int_{-1}^{+1}\mathrm{d}\cos\varphi\int_{-1}^{+1}\mathrm{d}\cos\psi\,\frac{T_{\text{irr}}(s;\cos\varphi)\,T(s;\cos\psi)}{\sqrt{-k(\cos\varphi,\cos\psi,\cos\theta)}}\,\theta(-k(\cos\varphi,\cos\psi,\cos\theta))\,.$$
$$-1\leqslant\cos\theta\leqslant+1\,,\qquad s\geqslant4m^2\,.$$

It is

$$(6.3)\qquad T_{\text{irr}}(k_1k_2\,|\,k_1'k_2') = (\Phi^{\text{in}}_{k_1'k_2'},\,K\Phi^{\text{in}}_{k_1k_2})\qquad\text{for } 4m^2\leqslant s<16m^2,$$

where the K-matrix is as usual defined by

$$K = 2i\,\frac{1-S}{1+S}\,.$$

As T and T_{irr} are boundary values of analytical functions in s both sides of this equation can be continued analytically to

$$(6.4)\qquad \Phi(s;\cos\theta) = \Phi_{\text{irr}}(s;\cos\theta) +$$
$$-\frac{i}{8}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\int_{-1}^{+1}\mathrm{d}\cos\varphi\int_{-1}^{+1}\mathrm{d}\cos\psi\,\frac{\Phi_{\text{in}}(s;\cos\varphi)\,\Phi(s;\cos\psi)}{\sqrt{-k(\cos\varphi,\cos\psi,\cos\theta)}}\,\theta(-k(\cos\varphi,\cos\psi,\cos\theta))\,.$$

This differs from the integral eq. (4.2) for the scattering amplitude on the second sheet only by a factor of two in the second term. Therefore the same method as in Section **4** can be applied for continuing $\Phi_{\text{irr}}(s;\cos\theta)$ into the complex $\cos\theta$-plane. As result we obtain that $\Phi_{\text{irr}}(s;\cos\theta)$ has the same analytic properties as the scattering amplitude on the second sheet (see Section **4**·3) The normal s-cut, however, starts first at $s=16m^2$ and the poles (C) are located differently for Φ_{irr}.

It follows that the continuation Φ_{irr} of the K-matrix element does not satisfy the Mandelstam representation or the usual dispersion relations for fixed momentum transfer.

7. – Crossing symmetry.

The analytic behavior of the scattering amplitude at $t=4m^2$ and $u=4m^2$ is given by

$$(7.1)\qquad \Phi(t,u) = F(t,u) + i\,\frac{\pi}{4}\frac{\sqrt{t-4m^2}}{\sqrt{s}}\,G(t,u)\,,$$

$$(7.2)\qquad \Phi(u,s) = F(u,s) + i\,\frac{\pi}{4}\frac{\sqrt{u-4m^2}}{\sqrt{s}}\,G(u,s)\,,$$

as immediately follows by applying crossing symmetry to (3.12). In this section we will derive a formula which gives the analytic continuation on the Riemann surfaces $\sqrt{s-4m^2}$, $\sqrt{t-4m^2}$ and $\sqrt{u-4m^2}$ simultaneously. We first define a crossing symmetric function R by

$$(7.3)\qquad R(s,t)=\Phi(s,t)-i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G(s,t)-$$
$$-i\frac{\pi}{4}\frac{\sqrt{t-4m^2}}{\sqrt{t}}G(t,u)-i\frac{\pi}{4}\frac{\sqrt{u-4m^2}}{\sqrt{u}}G(u,s)\,.$$

In the elastic region of s we have (except for $t\leqslant 0$, $u\leqslant 0$)

$$\lim_{\varepsilon\to+0}\big(R(s+i\varepsilon,t)-R(s-i\varepsilon,t)\big)=0\qquad\text{if }4m^2\leqslant s<16m^2$$

because

$$\lim_{\varepsilon\to+0}\big(\Phi(s+i\varepsilon,t)-\Phi(s-i\varepsilon,t)\big)=i\frac{\pi}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G(s,t)\,,$$
$$\text{for }4m^2\leqslant s<16m^2\,.$$
$$\lim_{\varepsilon\to+0}\big(G(t,s+i\varepsilon)-G(t,s-i\varepsilon)\big)=0\,.$$

Because of crossing symmetry $R(s,t)$ is also continuous across $4m^2\leqslant t<16m^2$ and $4m^2\leqslant u<16m^2$.

Hence $R(s,t)$ is analytic in s and t except for

(*a*) the normal cuts $s\geqslant 16m^2$, $t\geqslant 16m^2$, $u\geqslant 16m^2$;

(*b*) the kinematical cuts $s\leqslant 0$, $t\leqslant 0$, $u\leqslant 0$;

(*c*) poles of the form

$$\sum_\alpha\frac{r_{a\alpha}(1+2t/(s-4m^2))}{(s-a)^{n_\alpha}}\,,\qquad\sum_\alpha\frac{r_{a\alpha}(1+2u/(t-4m^2))}{(t-a)^{n_\alpha}}\,,\qquad\sum_\alpha\frac{r_{a\alpha}(1+2s/(u-4m^2))}{(u-a)^{n_\alpha}}\,.$$

the coefficients $r_{a\alpha}(x)$ are polynomials in x.

(*d*) Landau singularities and cuts coming from $G(s,t)$, $G(r,u)$ and $G(u,s)$.

As R is regular at $s=4m^2$, $t=4m^2$ and $u=4m^2$ the formula

$$(7.4)\qquad\Phi(s,t)=R(s,t)+$$
$$+i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}G(s,t)+i\frac{\pi}{4}\frac{\sqrt{t-4m^2}}{\sqrt{t}}G(t,u)+i\frac{\pi}{4}\frac{\sqrt{u-4m^2}}{\sqrt{u}}G(u,s)\,,$$
$$u=4m^2-s-t\,,$$

defines Φ simultaneously on the Riemann surfaces $\sqrt{s-4m^2}$, $\sqrt{t-4m^2}$ and $\sqrt{u-4m^2}=i\sqrt{s+t}$. If s and t lie on the first sheet of the Riemann surface $\sqrt{s-4m^2}$ the kinematical cuts, poles and Landau singularities of $G(s,t)$ are compensated by $R(s,t)$ in (4). Correspondingly if s and t lie on the first sheet of $\sqrt{t-4m^2}$ or $\sqrt{u-4m^2}$.

* * *

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APPENDIX

For the equal mass case Mandelstam has proved some analytic properties of the scattering amplitude in both variables [24]. As consequence of the dispersion relations and the regularity in the Lehmann ellipse the scattering amplitude $\Phi(s;\cos\theta)$ is regular in the domain

$$|s(s-4m^2)^2(1-\cos^2\theta)|=4|stu|<4A\,, \tag{A.1}$$

except for the cuts

$$s\leqslant 0\,,\qquad s\geqslant 4m^2\,, \tag{A.2}$$

$$\cos\theta=\pm\left(1+\frac{2t}{s-4m^2}\right)\,.\qquad t\geqslant 4m^2\,. \tag{A.3}$$

In case of a pair theory Mandelstam obtained $A=7\,168m^6$.

The partial wave amplitudes $\Phi_l(s)$ are regular inside

$$|s(s-4m^2)^2|<4A\,, \tag{A.4}$$

except for the cuts $s\leqslant 0$, $s\geqslant 4m^2$. (4) is a bounded domain of the complex s-plane and contains $0\leqslant s\leqslant 16m^2$.

By $E(a)$ we will denote an ellipse of the complex $\cos\theta$-plane with foci ± 1 and semi-major axis $a>1$. For s satisfying (4) one can inscribe an ellipse $E(a_l(s))$ into the domain (1) of the $\cos\theta$-plane. The semi-major axis is given by

$$a_l(s)=\sqrt{\frac{4A}{|s(s-4m^2)^2|}}\,. \tag{A.5}$$

([24]) S. MANDELSTAM: *Nuovo Cimento*, **15**, 658 (1960).

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The ellipse $E(a_i(s))$ touches the domain (1) at the ends of the minor axis. If s approaches the boundary

$$\text{(A.6)} \qquad |s(s-4m^2)^2| = 4A\,,$$

of (4) we have $a_i(s) \to 1$.

$\Phi(s; \cos\theta)$ is analytic in $\cos\theta$ if s belongs to (2) and $\cos\theta$ to $E(a_i(s))$ except for the cuts (3). These cuts lie completely outside the ellipse $E(a_i(s))$ if s is close enough to the boundary (6). For sufficiently small values of $|s|$ the cuts start inside $E(a_i(s))$. The starting points of the cuts lie on the ellipse $E(a_1(s))$ with

$$\text{(A.7)} \qquad a_1(s) = \sqrt{1 + 8m^2\,\frac{|s| + \operatorname{Re} s}{|s-4m^2|^2}}\,.$$

The partial wave expansion of $\Phi(s, \cos\theta)$ converges in

$$\text{(A.8)} \qquad \left\{\begin{array}{ll} E(a_i(s)) & \text{for} \quad a_1(s) > a_i(s)\,, \\ \text{and in} & \\ E(a_1(s)) & \text{for} \quad a_1(s) < a_i(s) \text{ and } s \gtrless 0\,, \end{array}\right.$$

provided that s belongs to (4). These are Mandelstam's results about the analytic properties of the scattering amplitude on the first sheet.

The results of this paper can now easily be established within the restrictions imposed by (1). As in Section **3** the functions $\Phi_l^{\mathrm{irr}}(s)$, $F_l(s)$, $G_l(s)$ can be defined as analytic functions of s regular in (4) except for poles and the cuts $s \leqslant 0$, $s \geqslant 16m^2$. All results of Section **2** are valid inside the domain (4).

The partial wave expansion (3.4), (3.5) of $\Phi_{\mathrm{irr}}(s; \cos\theta)$, $F(s; \cos\theta)$ converge inside

$$E(a_i(s))\,, \qquad \text{for } a_1(s) > a_i(s)\,,$$

$$E(a_1(s))\,, \qquad \text{for } a_1(s) < a_i(s):$$

The expansion (3.6) of $G(s; \cos\theta)$ converges inside

$$E(2a_i^2 - 1)\,, \qquad \text{for } a_1(s) > a_i(s)\,,$$

$$E(2a_1^2 - 1)\,, \qquad \text{for } a_1(s) < a_i(s)\,.$$

If $a_1 < a_i$ the method of Section **4** can be applied in order to continue Φ_{irr}, F, G and $\Phi^{(2)}$ beyond the convergence domain of the partial wave expansions. We will give the final results only. Let s be a point of the region (4) satisfying

$$a_1(s) < a_i(s) \qquad \text{and} \qquad s \lessgtr 0\,.$$

We will use the notation (4.5) and

$$a_i(s) = \cosh\alpha_i(s)\,.$$

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If n is the integer determined by

$$(n-1)\,\mathrm{Re}\,\alpha < \alpha_i < n\,\mathrm{Re}\,\alpha\,,$$

the function $G(s;\cos\theta)$ can be continued into the ellipse

$$E(a_1 a_i + \sqrt{a_1^2-1}\sqrt{a_i^2-1}) = E(\cosh(\mathrm{Re}\,\alpha_1 + \alpha_i))\,,$$

except for the domain $D_1^{\pm}, \ldots, D_n^{\pm}$. (The same regularity domains follow for Φ_{irr} by using Heitler's integral equation.) The pole terms can be treated as in Section **4**·6.

For any s considered only a finite number n of Landau singularities (5.1) appear in the domain $E(\cosh(\alpha_1+\alpha_i))$. But for $|s| \to 0$ we have $n \to \infty$, *i.e.* any of the Landau singularities can be reached by analytic continuation if $|s|$ is chosen sufficiently small.

RIASSUNTO (*)

Si mostra che l'ampiezza di scattering ha un branching a due foglietti nel punto di energia cinetica zero. Si discute dettagliatamente la continuazione analitica nel secondo foglietto.

(*) *Traduzione a cura della Redazione.*

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The Supplementary Condition in Quantum Electrodynamics.

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Summary. — A presentation of the supplementary condition in quantum electrodynamics is given which is free from normalization difficulties even with a definite metric. The supplementary condition is treated as a generator of a commutative sub-algebra in the ring of field operators. This sub-algebra generates an ideal in its own commutator algebra, and the corresponding remainder ring is the set of all physical gauge invariant operators. Any representation of this remainder ring is free from normalization difficulties. The relationship of this approach to that of Fermi, Candlin and Gupta (1) is discussed.

1. – Introduction.

Although the quantization of Maxwell's equations is not as straightforward a procedure as that of, say the scalar Klein-Gordon equation, and although the introduction of the supplementary condition has been presented in various ways, it would be fair to say that there is no great difficulty in principle involved. The apparent appearance of negative energies in the quantization of the free field is soon mitigated by the exact cancellation introduced by the supplementary condition. However, the usual procedure, following Fermi, of assuming that the physical state of the system is described by a state vector confined to a subspace of the total space of all possible electromagnetic states, leads to the conclusion that all the physical states are non-normalizable (2). One process for evading this difficulty is to introduce the indefinite metric

(1) E. Fermi: *Rev. Mod. Phys.*, **4**, 125 (1932); D. J. Candlin: *Nuovo Cimento*, **12**, 54 (1959); S. N. Gupta: *Proc. Phys. Soc.*, A **63**, 681 (1950); K. Bleuler: *Helv. Phys. Acta*, **23**, 567 (1950).

(2) S. T. Ma: *Phys. Rev.*, **75**, 535 (1949); **80**, 729 (1950); F. J. Belinfante: *Phys. Rev.*, **76**, 226 (1949).

interpretation of normalization of states, and this approach seems to be free of all contradiction.

Recently, CANDLIN (1) has shown that an alternative method would be to alter the supplementary condition in such a fashion that the physical consequences are unaffected, and the difficulty of normalization is removed, whilst a definite metric is retained. But the choice of an alternative supplementary condition is to some degree arbitrary, and this might be considered an unsatisfactory feature.

Several authors (3) have pointed out that it would be preferable to work with gauge invariant quantities from the beginning, and not to introduce the supplementary condition. But the procedure of quantization is more straightforward when expressed in terms of potentials.

In this paper a formulation of the quantum theory of the electromagnetic field will be given which is algebraic in standpoint. This approach enables a more unified treatment to be given and shows the relations between the various approaches. Because the ultimate aim is to enable the physical description in terms of gauge invariant quantities to be given in a consistent fashion, the physical content of quantum electrodynamics should be independent of the method of quantization, *i.e.*, whatever the superficial differences in the various theories, they have a common core.

In Section **2** the free field is discussed algebraically, and it will be shown that there is no difficulty of normalization, gauge invariance, or Lorentz covariance, if a certain ring of operators is considered.

In Section **3**, this method is extended to the electromagnetic field with sources (the case of only a single charged spinor field is considered for simpicity). and in Section **4**, the relationship of the various alternative approaches is described.

Most of the discussion will be given in the Schrödinger representation of operators, because, although the Heisenberg representation has the advantgae that the covariance of the theory is manifest, the fact that the algebraic properties of the operators is largely unknown detracts from its value. Whereas, for the Schrödinger representation, both the algebraic and Lorentz transformation properties may be exhibited explicitly, although, in general, not in so symmetric a form.

The units and notation will be the same as in Wentzel, with greek indices running over 1, 2, 3, 4 and latin indices over 1, 2, 3. The complex conjugate of a number, and Hermitean conjugate of an operator will be denoted by a bar. The three dimensional scalar product is denoted by $\boldsymbol{a}\cdot\boldsymbol{b}$, and the four dimensional product by ab.

(3) W. THIRRING: *Einführung in die Quantenelektrodynamik* (Wien, 1955); J. HAMILTON: *Theory of Elementary Particles* (Oxford, 1959).

2. – Free field case.

We consider a family of operators $q_\mu(\boldsymbol{k})$, $p_\mu(\boldsymbol{k})$ with $\boldsymbol{k}$ ranging over three dimensional momentum space. These operators are restricted by the commutation relations

$$[q_\mu(\boldsymbol{k}), p_{\mu'}(\boldsymbol{k}')] = i\hbar\delta_{\mu\mu'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,, \tag{2.1}$$

and the reality conditions

$$\begin{cases} \overline{q_j(\boldsymbol{k})} = q_j(-\boldsymbol{k})\,, & \overline{p_j(\boldsymbol{k})} = p_j(-\boldsymbol{k})\,, \\ \overline{q_4(\boldsymbol{k})} = -q_4(-\boldsymbol{k})\,, & \overline{p_4(\boldsymbol{k})} = -p_4(-\boldsymbol{k})\,, \end{cases} \tag{2.2}$$

so that the Fourier transforms

$$A_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3k\, q_\mu(k)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,, \qquad \Pi_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3k\, p_\mu(k)\exp[-i\boldsymbol{k}\cdot\boldsymbol{x}]\,,$$

are Hermitean for $\mu = 1\,2,3$ and anti Hermitean for $\mu = 4$.

Because of the δ-functions in the commutation relations, these operators are singular, but from them well defined operators

$$q_\mu(F) = \int \mathrm{d}^3k\, F(\boldsymbol{k})q_\mu(\boldsymbol{k})\,, \qquad p_\mu(F) = \int \mathrm{d}^3k\, F(\boldsymbol{k})p_\mu(k)\,,$$

can be derived such that, if $F(\boldsymbol{k})$ is a sufficiently smooth testing function, the commutation relations are no longer singular. With this understanding, the operators $q_\mu(\boldsymbol{k})$, $p_\mu(\boldsymbol{k})$ will be used in the following discussion.

Such a family of operators generates an irreducible algebra of operators $\mathscr{A}$ on a Hilbert space. The classification of possible representations has been given by WIGHTMAN and SCHWEBER [4]. On this ring of operators a family of inner automorphisms is defined which is an operator representation of the inhomogeneous Lorentz group. The infinitesimal time translation is expressed as a system of equations of motion. For the free electromagnetic field these equations of motion are

$$\begin{cases} \dot{A}_j = c^2\Pi_j + ic\,\dfrac{\partial A_4}{\partial x_j}\,, & \dot{A}_4 = c^2\Pi_4 - ic\,\dfrac{\partial A_j}{\partial x_j}\,, \\ \dot{\Pi}_j = -\dfrac{\partial^2 A_l}{\partial x_j\partial x_l} + \dfrac{\partial^2 A_j}{\partial x_l^2} - ic\,\dfrac{\partial \Pi_4}{\partial x_j}\,, & \dot{\Pi}_4 = ic\,\dfrac{\partial \Pi_j}{\partial x_j}\ (^*)\,. \end{cases} \tag{2.3}$$

(4) A. S. WIGHTMAN and S. S. SCHWEBER: *Phys. Rev.*, **98**, 812 (1955).

(*) The eq. (2.3) are those derivable from the Fermi Lagrangian. The alternative Lagrangian density $-\frac{1}{2}(\partial A_\nu/\partial x_\mu)^2$ is obtained by making the canonical transformation

$$G = \exp\left(+\frac{i}{\hbar c}\int \mathrm{d}^3x_4 A(x)\,\frac{\partial A_i}{\partial x_i}\right).$$

Corresponding equations for the Fourier transforms can be readily derived.

The solution of (2.3) can be given in terms of annihilation and creation operators $a_\mu(\boldsymbol{k})$, $\overline{a_\mu}(\boldsymbol{k})$ by:

$$
(2.4)\qquad\left\{
\begin{aligned}
&q_j(\boldsymbol{k}) = \sqrt{\frac{\hbar c}{2k}}\,\big(a_j(\boldsymbol{k}) + \overline{a_j(-\boldsymbol{k})}\big)\,, \qquad q_4(\boldsymbol{k}) = -\sqrt{\frac{\hbar c}{2k}}\,\big(a_4(\boldsymbol{k}) - \overline{a_4(\boldsymbol{k})}\big)\,,\\
&p_j(\boldsymbol{k}) = -\,i\sqrt{\frac{\hbar k}{2c}}\,\big(a_j(-\boldsymbol{k}) - \overline{a_j(\boldsymbol{k})}\big) + k_j\sqrt{\frac{\hbar}{2kc}}\,\big(a_4(-\boldsymbol{k}) - \overline{a_4(\boldsymbol{k})}\big)\,,\\
&p_4(\boldsymbol{k}) = -\,i\sqrt{\frac{\hbar k}{2c}}\,\big(a_4(-\boldsymbol{k}) + \overline{a_4(\boldsymbol{k})}\big) + k_j\sqrt{\frac{\hbar}{2kc}}\,\big(a_j(-\boldsymbol{k}) + \overline{a_j(\boldsymbol{k})}\big)\,.
\end{aligned}
\right.
$$

with the commutation relations

$$
(2.5)\qquad [a_\mu(\boldsymbol{k}), \overline{a_{\mu'}(\boldsymbol{k})}] = \delta_{\mu\mu'}\delta(\boldsymbol{k}-\boldsymbol{k}')\,.
$$

These operators satisfy the simple equations of motion

$$
(2.6)\qquad\left\{
\begin{aligned}
&\dot a_j(\boldsymbol{k}) = -\,ikc\,a_j(\boldsymbol{k})\,, \qquad \dot a_4(\boldsymbol{k}) = \ \ ikc\,\dot a_4(\boldsymbol{k})\,,\\
&\dot{\overline{a_j(\boldsymbol{k})}} = +\,ikc\,a_j(\boldsymbol{k})\,, \qquad \dot a_4(\boldsymbol{k}) = -\,ikc\,a_4(\boldsymbol{k})\,.
\end{aligned}
\right.
$$

The different behaviour of the time like components compared with the spacelike ones is manifest. The eq. (2.3), (2.6) are generated by a Hamiltonian whose Fourier component is

$$
(2.7)\qquad H(\boldsymbol{k}) = \tfrac{1}{2}\hbar kc\big(a_j(\boldsymbol{k})a_j(\boldsymbol{k}) + a_j(\boldsymbol{k})a_j(\boldsymbol{k})\big) - \big(a_4(\boldsymbol{k})a_4(\boldsymbol{k}) + a_4(\boldsymbol{k})a_4(\boldsymbol{k})\big)\,.
$$

In the usual (Fermi) approach, the state vectors are those vectors of the representation Hilbert space of the ring $\mathscr{A}$ which satisfy the supplementary conditions

$$
(2.8)\qquad \chi(x)\Psi = \dot\chi(x)\Psi = 0
$$

with

$$
\chi(x) = -\,ic\Pi_4(x)\,, \qquad \dot\chi(x) = -\,ic\dot\Pi_4(x) = c^2\,\frac{\partial\Pi_j}{\partial x_j}\,.
$$

The conditions (2.8) hold for each x, as $\chi(x)$ and $\dot\chi(x)$ form a family of commuting (singular) operators. An equivalent set of conditions is found by taking the Fourier transforms of $\chi(x)$ and $\dot\chi(x)$ and forming the combinations

$$
\chi^+(\boldsymbol{k}) = \frac{1}{c}\,\dot\chi(\boldsymbol{k}) - ik\chi(\boldsymbol{k})\,, \qquad \chi^-(\boldsymbol{k}) = \frac{1}{c}\,\dot\chi(\boldsymbol{k}) + ik\chi(\boldsymbol{k})\,,
$$

which can be written as

$$(2.9)\qquad \begin{cases} \chi^+(\boldsymbol{k}) = -ic\overline{k_\mu}\,p_\mu(\boldsymbol{k}) = -\overline{k_\mu}A_\mu(-\boldsymbol{k})\,, \\ \chi^-(\boldsymbol{k}) = -ick_\mu p_\mu(\boldsymbol{k}) = \quad k_\mu A^*_\mu(\boldsymbol{k})\,, \end{cases}$$

and they are the positive and negative frequency parts respectively of $\chi(\boldsymbol{k})$. The four-vector $A_\mu(\boldsymbol{k})$ is defined by $A_j(\boldsymbol{k}) = (2hkc)^{\frac{1}{2}}a_j(\boldsymbol{k})$, $A_4(\boldsymbol{k}) = (2hkc)^{\frac{1}{2}}a_4(-\boldsymbol{k})$, with $A^*_j(\boldsymbol{k}) = A_j(\boldsymbol{k})$, $A^*_4(\boldsymbol{k}) = -A_4(\boldsymbol{k})$. These four-vectors have the equations of motion

$$(2.6')\qquad \dot{A}_\mu(\boldsymbol{k}) = -ikcA_\mu(\boldsymbol{k})\,,\qquad \dot{A}^*_\mu(\boldsymbol{k}) = ikcA^*_\mu(\boldsymbol{k})\,.$$

As $\overline{\chi^+(\boldsymbol{k})} = \chi^-(-\boldsymbol{k})$, $[\chi^+(\boldsymbol{k}), \chi^+(\boldsymbol{k})] = 0$, so that $\chi^+(\boldsymbol{k})$ is a normal operator, and its eigenstates are identical with those of its Hermitean conjugate. So the conditions corresponding to (2.9) are not independent.

The inner automorphisms corresponding to the representations of the inhomogeneous Lorentz group are generated by the infinitesimal transformations P_μ and $J_{\mu\nu}$, which, being elements of $\mathscr{A}$ are expressible in terms of A_μ and Π_μ, or, equivalently, in terms of q_μ and p_μ. Explicit expressions for P_μ and $J_{\mu\nu}$ can be obtained from the formulae of Schwinger [5], using the equations of motion to express $\dot{q}_\mu$ and $\dot{p}_\mu$ in terms of q_μ and p_μ on the hyperplane $x_4=0$. The transformations of the homogeneous group with parameters $\varepsilon_{\mu\nu}$ are given by

$$(2.10)\qquad \left|\begin{aligned}
(q_j(\boldsymbol{k}))' &= q_j(\boldsymbol{k}) - \varepsilon_{rs}k_s\frac{\partial q_j(\boldsymbol{k})}{\partial k_r} + \varepsilon_{jr}q_r(\boldsymbol{k}) + \varepsilon_{r4}c\frac{\partial p_j(-\boldsymbol{k})}{\partial k_r} - \varepsilon_{r4}k_j\frac{\partial q_4(\boldsymbol{k})}{\partial k_r}\,,\\
(q_4(\boldsymbol{k}))' &= q_4(\boldsymbol{k}) - \varepsilon_{rs}k_s\frac{\partial q_4(\boldsymbol{k})}{\partial k_r} + \varepsilon_{r4}c\frac{\partial p_4(-\boldsymbol{k})}{\partial k_r} + \varepsilon_{r4}k_j\frac{\partial q_j(\boldsymbol{k})}{\partial k_r}\,,\\
(p_j(\boldsymbol{k}))' &= p_j(\boldsymbol{k}) - \varepsilon_{rs}k_s\frac{\partial p_j(\boldsymbol{k})}{\partial k_r} + \varepsilon_{jr}p_r(\boldsymbol{k}) - \varepsilon_{r4}\frac{1}{c}\frac{\partial}{\partial k_r}(k_jk_rq_4(-\boldsymbol{k}) - k^2q_j(\boldsymbol{k})) +\\
&\qquad + \varepsilon_{r4}\frac{\partial}{\partial k_r}(k_jp_4(-\boldsymbol{k})) - \varepsilon_{r4}\frac{1}{c}(k_rq_j(-\boldsymbol{k}) - k_jq_r(-\boldsymbol{k}))\,,\\
(p_4(\boldsymbol{k}))' &= p_4(\boldsymbol{k}) - \varepsilon_{rs}k_s\frac{\partial p_4(\boldsymbol{k})}{\partial k_r} - \varepsilon_{r4}\frac{\partial}{\partial k_r}(k_jp_j(\boldsymbol{k}))\,.
\end{aligned}\right.$$

Under the transformation (2.10) it can be shown that

$$(2.11)\qquad (\chi^+(\boldsymbol{k}))' = \chi^+(\boldsymbol{k}')\,,\qquad (\chi^-(\boldsymbol{k}))' = \chi^-(\boldsymbol{k}'')\,,$$

[5] J. Schwinger: *Phys. Rev.*, 82, 914 (1951), especially eq. (2.92) and (2.96).

where $\boldsymbol{k}'$ is the spatial part of the Lorentz transformation of the four-vector k_μ and $\boldsymbol{k}''$ that with respect to the four-vector $\overline{k_\mu}$. From (2.11), it can be seen that $\chi^+(\boldsymbol{k})$, $\chi^-(\boldsymbol{k})$ form two separate invariant families of operators, whereas the sets $\chi(\boldsymbol{k})$, $\dot{\chi}(\boldsymbol{k})$ are not separately invariant, although the combined set must be so. So the conditions $\chi^+(\boldsymbol{k})\Psi = 0$ or $\chi^-(k)\Psi = 0$ can be imposed separately without violating Lorentz invariance. The eq. (2.6′) show that (2.9) are invariant (to within a numerical factor) under time translations.

The transformation formulae (2.10), whilst linear, are not simple, and a simpler form is found by taking the sets $A_\mu(\boldsymbol{k})$ or $A^*_\mu(\boldsymbol{k})$, for these transform as a four-vector:

$$(A_\mu(\boldsymbol{k}))' = A_\mu(\boldsymbol{k}') + \varepsilon_{\mu\nu} A_\nu(\boldsymbol{k}) ,$$

$$(A^*_\mu(\boldsymbol{k}))' = A^*_\mu(\boldsymbol{k}') + \varepsilon_{\mu\nu} A^*_\nu(\boldsymbol{k}) .$$

The operator algebra $\mathscr{A}$ is a $*$-algebra (v. Neumann or symmetric algebra [6]), *i.e.* if an operator $O \in \mathscr{A}$ then $\bar{O} \in \mathscr{A}$. The heuristic definition of $\mathscr{A}$ is as the set of all bounded operators which are formed from sums and products of the generators p_μ, q_μ, and from limiting operations performed upon these sums and products, using the definite metric of the representation space. More precisely, the ring $\mathscr{A}'$ of *bounded operators* which commute with all the q_μ and p_μ is formed (if the representation of these operators is irreducible, as may be assumed, $\mathscr{A}'$ consists of multiples of the identity only), and then the ring of bounded operators $\mathscr{A}''$ which commute with all the elements of $\mathscr{A}'$ is formed. $\mathscr{A}''$ is then the ring generated by q_μ and p_μ, and if the latter are irreducible, $\mathscr{A}''$ will contain all bounded operators. We will simply speak of the ring $\mathscr{A}$ instead of $\mathscr{A}''$, for simplicity.

The algebras $\mathscr{B}(\chi^+)$, $\mathscr{B}(\chi^-)$ generated by the families $\chi^+(\boldsymbol{k})$, $\chi^-(\boldsymbol{k})$ (for all $\boldsymbol{k}$) are invariant commutative sub-algebras of $\mathscr{A}$ (because of the eq. (2.11)).

So $\mathscr{B}(\chi^+) \equiv \mathscr{B}'(\chi^+)$, $\mathscr{B}(\chi^-) \equiv \mathscr{B}'(\chi^-)$.

The commutator algebras $\mathscr{C}(\chi^+)$, $\mathscr{C}(\chi^-)$ of operators which commute with all elements of $\mathscr{B}(\chi^+)$ and $\mathscr{B}(\chi^-)$ respectively are defined by $\mathscr{C}(\chi^+) \equiv \mathscr{B}'(\chi^+)$, $\mathscr{C}(\chi^-) \equiv \mathscr{B}(\chi^-)$. Clearly $\mathscr{B}(\chi^+) \subset \mathscr{C}(\chi^+)$ and $\mathscr{B}(\chi^-) \subset \mathscr{C}(\chi^-)$. The intersection $\mathscr{C}(\chi^+) = \mathscr{C}(\chi^-) \cap \mathscr{C}(\chi^-)$ is an invariant sub-algebra of $\mathscr{A}$ containing both $\mathscr{B}(\chi^-)$ and $\mathscr{B}(\chi^-)$. $\mathscr{C}(\chi)$ will be a $*$-algebra, because, if $O \in \mathscr{C}(\chi)$, then

$$[O, \mathscr{B}(\chi^+)] = [O, \mathscr{B}(\chi^-)] = 0 ,$$

and

$$[\bar{O}, \mathscr{B}(\chi^+)] = [\bar{O}, \mathscr{B}(\chi^-)] = 0 , \qquad [\bar{O}, \overline{\mathscr{B}(\chi^-)}] = [\bar{O}, \mathscr{B}(\chi^+)] = 0 ,$$

so that $\bar{O} \in \mathscr{C}(\chi)$ also.

([6]) J. Dixmier: *Les Algèbres d'Opérateurs dans L'Espace Hilbertien*, (Paris, 1957); M. A. Naimark: *Normed Rings* (Groningen, 1959).

$\mathscr{C}(\chi^+)$ is generated by the (not linearly independent) operators $\chi^+(\boldsymbol{k})$, $\chi^-(\boldsymbol{k})$, $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho p_\sigma(k)$, $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho q_\sigma(k)$. ($\chi^-(\boldsymbol{k}) \in \mathscr{B}(\varepsilon_{\mu\nu\varrho\sigma}k_\varrho p_\sigma)$ because $k_\mu p_\mu = (1/k_\varkappa)\varepsilon_{\varkappa\lambda\mu\nu}\varepsilon_{\mu\nu\varrho\sigma}k_\lambda k_\varrho p_\sigma$). There are seven linearly independent operators for each $\boldsymbol{k}$ in this set of generators. These generators are essentially the supplementary condition operators, together with the positive frequency components $F^+_{\mu\nu}(\boldsymbol{k})$ of the electromagnetic field operators. $\mathscr{A}$ can be generated from $\mathscr{C}(\chi^+)$ by adjoining the operator $k_\mu q_\mu(\boldsymbol{k})$. The algebra generated by $k_\mu q_\mu(\boldsymbol{k})$ is not an invariant algebra. Similarly $\mathscr{C}(\chi^-)$ is generated by χ^+, χ^-, $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho p_\sigma$ and $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho q_\sigma$. In terms of annihilation and creation operators $\mathscr{C}(\chi^+)$ is generated by $k_\mu A_\mu(\boldsymbol{k})$, $k_\mu A^*_\mu$, $\overline{k_\mu}A_\mu$, $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho A_\sigma$ and $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho A^*_\sigma$, whilst $\mathscr{C}(\chi^-)$ is generated by χ^+, χ^-; $\overline{k_\mu}A^*_\mu$, $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho A_\sigma$, and $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho A^*_\sigma$. $\mathscr{C}(\chi)$ is generated by χ^+, χ^-, $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho A_\sigma$ and $\varepsilon_{\mu\nu\varrho\sigma}k_\varrho A^*_\sigma$. $\mathscr{A}$ is generated from $\mathscr{C}(x)$ by adjoining the two non-invariant families $\overline{k_\varrho}A_\varrho(\boldsymbol{k})$ and $\overline{k_\varrho}A^*_\varrho(\boldsymbol{k})$.

The group of Lorentz transformations $\mathscr{L}$ are defined by $\mathscr{L}A = LAL^{-1}$, where L is an element of the operator representation of $\mathscr{L}$. From the discussion already given $\mathscr{L}\mathscr{B} = \mathscr{B}$, $\mathscr{L}\mathscr{C} = \mathscr{C}$. As $\mathscr{L}\mathscr{B} = \mathscr{B}$, we have $LBL^{-1} = B' \in \mathscr{B}$, for all $B \in \mathscr{B}$. So $LB = B'L$ and $L \notin \mathscr{C}$, unless $B' = B$, *i.e.* those L which commute with all B. But the only L for which this is so is the identity. Hence $\mathscr{L}\mathscr{C}$ is an *outer automorphism* of $\mathscr{C}$.

Any operator contained in $\mathscr{C}$ is a Lorentz gauge invariant operator. A general gauge transformation is a member of the family of unitary transformations U generated by the operators

$$
(2.12)\quad
\begin{cases}
T(f, t) = -\dfrac{1}{hc}\displaystyle\int \mathrm{d}^3k\left(f^+(\boldsymbol{k}, t)\,k_\mu A_\mu(-\boldsymbol{k}, t) - f^-(\boldsymbol{k}, t)k_\mu A_\mu(\boldsymbol{k}, t)\right), \\[2ex]
S(g, t) = \dfrac{1}{4hc}\displaystyle\int \dfrac{\mathrm{d}^3k}{k^3}\left(g^+(\boldsymbol{k}, t)\,k_\mu A_\mu(\boldsymbol{k}, t) + g^-(\boldsymbol{k}, t)\,k_\mu A_\mu(\boldsymbol{k}, t)\right),
\end{cases}
$$

where $f^+(\boldsymbol{k}, t)$, $f^-(\boldsymbol{k}, t)$, $g^+(\boldsymbol{k}, t)$, $g^-(\boldsymbol{k}, t)$ are arbitrary functions, restricted by the conditions that T and S are antihermitean. This will be so if $\overline{f^+(+\boldsymbol{k}, t)} = -f^-(-\boldsymbol{k}, t)$, $g^+(\boldsymbol{k}, t) = -g^-(\boldsymbol{k}, t)$. As $k_\mu A_\mu(\boldsymbol{k}, t)$ and $k_\mu A^*_\mu(\boldsymbol{k}, t)$ are elements of $\mathscr{B}(\chi^+)$ and $\mathscr{B}(\chi^-)$ respectively, $T \in \mathscr{C}$, but $S \notin \mathscr{C}$. Under the transformation

$$
(2.13)\quad
\begin{cases}
O' = O + [T, O] + [S, O], \\[1ex]
A_\mu(\boldsymbol{k}, t) \to A_\mu - 2kf^-(\boldsymbol{k}, t) - \dfrac{k_\mu}{2k^2}\,g^+(\boldsymbol{k}, t).
\end{cases}
$$

If this is to be a gauge transformation, we must have

$$
\frac{\partial}{\partial t}\left(f^-(\boldsymbol{k}, t) + \frac{1}{4k^3}\,g^+(\boldsymbol{k}, t)\right) = -ikc\left(f^-(\boldsymbol{k}, t) - \frac{1}{4k^3}\,g^+(\boldsymbol{k}, t)\right),
$$

and then if we write

$$A(x,t)=\frac{i}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3k\left(f^-(\boldsymbol{k},t)+\frac{1}{4k^3}\,g^+(\boldsymbol{k},t)\right)\exp\left[i\boldsymbol{k}\cdot x\right]+\text{h.c.}\,,$$

the transformations (2.13) are equivalent to the transformation

$$A_\mu(x,t)\to A_\mu(x,t)+\frac{\partial\Lambda}{\partial x_\mu}\,, \tag{2.13'}$$

for the time dependent Heisenberg operators. The conditions

$$f^+(\boldsymbol{k},t)=-\,f^-(-\,\boldsymbol{k},t)\qquad\text{and}\qquad \overline{g^+(\boldsymbol{k},t)}=-\,g^-(\boldsymbol{k},t)\,,$$

are equivalent to the reality condition $\overline{A(x,t)}=A(x,t)$. If $g^+\equiv 0$, we have the transformations of the Lorentz gauge for which $\Box^2\Lambda=0$. Under (2.13), $k_\mu A_\mu\to k_\mu A_\mu-g^+(k,t)$, so that $\mathscr{B}(\chi^\pm)\to\mathscr{B}(\chi^\pm)$, and $\mathscr{C}$ is invariant.

Any operator $C\in\mathscr{C}$ commutes with any T, so that the algebra $\mathscr{C}$ is gauge invariant under Lorentz gauge transformations. So the representation of $\mathscr{C}$ on a Hilbert space is a field theory of the electromagnetic field which is automatically Lorentz gauge invariant:

As the elements $\chi^+(\boldsymbol{k})$, $\chi^-(\boldsymbol{k})$ commute with all the elements of $\mathscr{C}$, we can form a two sided ideal $\mathscr{I}(\chi^+,\chi^-)\subset\mathscr{C}$ consisting of all elements of the form $C^+\chi^++C^-\chi^-$ where C^+, $C^-\in\mathscr{C}$ (*).

Other ideals can be generated by the elements $\chi^\pm(\boldsymbol{k})-Eg^\pm(\boldsymbol{k})$, for arbitrary functions $g^\pm(\boldsymbol{k})$. E is the unit element. These ideals could be denoted by $\mathscr{I}_g(\chi^+,\chi^-)$, and with this notation $\mathscr{I}\equiv\mathscr{I}_0$. The ideal $\mathscr{I}_0$ generates a remainder ring $\mathscr{C}^L$, which is the description of the electromagnetic field in the Lorentz gauge. This means that $\mathscr{C}^L$ can be decomposed into classes generated by $\mathscr{I}_0$. If

$$\mathscr{C}=\mathscr{I}_0+(C_1)+(C_2)+\dots\,, \tag{2.14}$$

where: (C_1) is the set of elements $C_1+\mathscr{I}_0$, $C_1\notin\mathscr{I}_0$,
(C_2) is the set of elements $C_2+\mathscr{I}_0$, $C_2\notin\mathscr{I}_0$, (C_1) etc.,

then the mapping $\mathscr{G}$: $\mathscr{I}_0\to 0$, $(C_1)\to C_1'$, $(C_2)\to C_2'$, ... is a ring homomorphism.

(*) As χ^+, χ^- are unbounded operators, they do not, strictly, belong to $\mathscr{I}$, so that $\mathscr{I}\not\subset\mathscr{C}$. But instead of the operators themselves, the truncated operators $[\chi^+]$, $[\chi^-]$ such that

$$[\chi^+]\,\Phi_\lambda=\chi^+\Phi_\lambda=\lambda\Phi_\lambda\ \text{for}\ |\lambda|\leqslant\lambda_0\,,\quad\text{and}\quad[\chi^+]\Phi_\lambda=0\ \text{for}\ |\lambda|>\lambda_0\,,$$

could be used.

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Essentially this means that all operators contained in $\mathscr{C}$ which have χ^- or χ^- (or $[\chi^+]$, $[\chi^-]$) as a factor are put equal to zero, *i.e.* the zero eigenvalue of χ^+ and χ^- for all $\boldsymbol{k}$ is taken. Under this homomorphism $\mathscr{L}C \to (\mathscr{L}C)' = \mathscr{L}'C'$, defining the mapping $\mathscr{L}'$. If $\mathscr{L}'$ is to be a homomorphism, it is necessary that

$$\mathscr{L}'(C_1C_2)' = (\mathscr{L}'C_1')(\mathscr{L}'C_2')$$

and this will be so because

$$\mathscr{L}'(C_1C_2)' = (\mathscr{L}C_1C_2)' = \big((\mathscr{L}C_1)(\mathscr{L}C_2)\big)' = (\mathscr{L}C_1)'(\mathscr{L}C_2)' = (\mathscr{L}'C_1')(\mathscr{L}'C_2') .$$

This is a consequence of the fact that the successive applications of two homomorphic mappings $\mathscr{L}$ and $\mathscr{G}$ is itself a homomorphic mapping $\mathscr{L}' = \mathscr{G}\mathscr{L}$. $\mathscr{L}$ is an inner automorphism of $\mathscr{A}$, and, although $\mathscr{I}_0$ is not in general an ideal of $\mathscr{A}$, it is an invariant subring. So

$$\mathscr{L}'C' = L(C)L^{-1} = L(C+\mathscr{I}_0)L^{-1} = LCL^{-1} + \mathscr{I}_0 = \mathscr{L}C + \mathscr{I}_0 = (\mathscr{L}C)'.$$

Although $\mathscr{L}\mathscr{C}$ is an outer automorphism of $\mathscr{C}$, $\mathscr{L}\mathscr{C}L$ is an *inner automorphism.* For, suppose T is an arbitrary infinitesimal Lorentz gauge transformation, and C is an arbitrary element of $\mathscr{C}$. Then because C is gauge invariant, and gauge invariance is preserved by Lorents transformations, we have

$$[T, C] = 0 = [T, [J_{\mu\nu}, C]] .$$

From the Jacobi identity.

$$[T, [J_{\mu\nu}, C]] + [J_{\mu\nu}, [C, T]] + [C, [T, J_{\mu\nu}]] = 0 \qquad \text{or} \qquad [C, [T, J_{\mu\nu}]] = 0 .$$

If C is chosen to be an arbitrary element of $\mathscr{B}$, then, as $[T, J_{\mu\nu}]$ consequently commutes with all elements of $\mathscr{B}$, it is contained in $\mathscr{C}$. But as it commutes with all elements of $\mathscr{C}$, it must be contained in $\mathscr{B}$. So, in particular we have

$$[k_\varrho A_\varrho, J_{\mu\nu}] = B^+_{\mu\nu}(\boldsymbol{k})$$

and

$$[k_\varrho A^*_\varrho, J_{\mu\nu}] = B^-_{\mu\nu}(\boldsymbol{k}) .$$

Then

$$J_{\mu\nu} - \int \frac{d^3k'}{4hck'^2}\, \overline{k'_\varrho}\big(B^+_{\mu\nu}(\boldsymbol{k}')A^*_\varrho(\boldsymbol{k}') - B^-_{\mu\nu}(\boldsymbol{k}')A_\varrho(\boldsymbol{k}')\big) ,$$

commutes with χ^+ and χ^-. Hence

$$J_{\mu\nu} = K_{\mu\nu} + \int \frac{d^3k'}{4hck'^2} \overline{k'_\varrho} (B^+_{\mu\nu}(\boldsymbol{k}') A^*_\varrho(\boldsymbol{k}') - B^-_{\mu\nu}(\boldsymbol{k}') A_\varrho(\boldsymbol{k}')) \,, \tag{2.15}$$

with $K_{\mu\nu} \in \mathscr{C}$. If we write $J_{\mu\nu} = K_{\mu\nu} + \widetilde{K}_{\mu\nu}$, then as $[J_{\mu\nu}, C]$ and $[K_{\mu\nu}, C]$ are both in $\mathscr{C}$, $[\widetilde{K}_{\mu\nu}, C] \in \mathscr{C}$ for all C. But as $[\overline{k'_\varrho} A^*_\varrho(\boldsymbol{k}'), C]$, $[\overline{k'_\varrho} A_\varrho(\boldsymbol{k}'), C] \in \mathscr{C}$ for all $C \in \mathscr{C}$, $[\widetilde{K}_{\mu\nu}, C] \in \mathscr{I}_0$ if $B^\pm_{\mu\nu}(\boldsymbol{k}) \in \mathscr{I}_0$. So if $B^+_{\mu\nu}(\boldsymbol{k}') \in \mathscr{I}_0$ the term $\widetilde{K}_{\mu\nu}$ in $J_{\mu\nu}$ makes no contribution when computing $\mathscr{L}'$ and may be dropped, *i.e.* $\mathscr{L}'$ is generated by $K_{\mu\nu}$, and so is an inner automorphism. This construction shows that, although the additional term $\widetilde{K}_{\mu\nu}$ is not contained in $\mathscr{I}_0$ (being formed from elements not in $\mathscr{C}$) it may be treated as if it were, so long as $B^\pm_{\mu\nu}(\boldsymbol{k}') \in \mathscr{I}_0$. This property of Lorentz transformations being inner automorphisms of $\mathscr{C}^L$ as well as of $\mathscr{A}$ is an important one, for it means that the algebra $\mathscr{C}^L$ is completely self contained, and actually makes no reference to external quantities when discussing its structure. This is the algebraic equivalent of the physical fact that the electromagnetic field theory may be constructed with no reference to potentials. Furthermore the arguments presented depend little on the detailed structure of the electromagnetic field and the supplementary condition, and so may be applied in other contexts. The essential requirements for such a theory of a supplementary condition in a relativistic field theory are:

a) a $*$-algebra of field operators which is irreducible, so that the Lorentz transformations are inner automorphisms,

b) an invariant family of normal operators, so that the commutator algebra is also a $*$-algebra,

and

c) Lorentz transformations are inner automorphisms of the remainder ring.

Condition *c*) is a consequence of *a*) and *b*) for electromagnetic theory as $B^\pm_{\mu\nu}(\boldsymbol{k}) \in \mathscr{I}_0$.

The inner automorphism of $\mathscr{A}$ generated by S carries $\chi^+(\boldsymbol{k})$ into $\chi^\pm(\boldsymbol{k}) - g^\pm(\boldsymbol{k}, t)E$, so that the ideal $\mathscr{I}_g$ is ismorphic with $\mathscr{I}_0$. Furthermore the remainder rings $\mathscr{C}^L$ and $\mathscr{C}^g$ are subrings of $\mathscr{A}$ and so are also isomorphic. In fact they are identical. This is the statement of invariance of the electromagnetic field under general gauge transformations and not merely under the Lorentz gauge transformations.

The remainder rings can be generated by operators which are linearly independent of $k_\mu p_\mu$, $\overline{k_\mu} p_\mu$, $k_\mu q_\mu$ and $\overline{k_\mu} q_\mu$ (or $k_\mu A_\mu$, $k_\mu A^*_\mu$, $\overline{k_\mu} A_\mu$ and $\overline{k_\mu} A^*_\mu$). Such operators, not a linearly independent set, are the component of the electromagnetic field $F_{\mu\nu}$, or alternatively $\varepsilon_{ijk} k_j p_k$ and $\varepsilon_{ijk} k_j q_k$. A linearly inde-

pendent set would be say, $k_1p_2 - k_2p_1$, $k_2p_3 - k_3p_2$, $k_1q_2 - k_2q_1$ and $k_2q_3 - k_3q_2$, which under the Lorentz transformation $K_{\mu\nu}$ transform amongst themselves, although not as simply as A_μ and A_μ^*. Various attempts using this idea have been made by NOVOBATSKY, VALATIN, JAUCH and COESTER, and SCHWINGER [7]. In the free field case these operators transform linearly and so the representation induced in them by the Lorentz group could be simplified by choosing suitable linear combinations.

The algebraic point of view advanced in this section can be put in a more familiar form by considering the operators $\chi^+(\boldsymbol{k})$ and $\chi^-(\boldsymbol{k})$ and their spectrum.

Because these operators have a continuous spectrum, states corresponding to eigenvalues $g^+(\boldsymbol{k})$ and $g^-(\boldsymbol{k})$ respectively are not normalizable. But the Hilbert space $\mathscr{H}$ in which these operators act can be expressed as a direct integral over the spectrum of $\chi^+(\boldsymbol{k})$ and $\chi^-(\boldsymbol{k})$.

We can write

$$\mathscr{H} = \iint \delta g^+(\boldsymbol{k})\, \delta g^-(\boldsymbol{k}) \oplus \mathscr{H}\big(g^+(\boldsymbol{k}), g^-(\boldsymbol{k})\big) \; (^*) , \tag{2.16}$$

where the integration is over all functions $g^+(\boldsymbol{k})$, and $\mathscr{H}(g^+, g^-)$ is a Hilbert space corresponding to each choice of spectral functions g^+ and g^-. So although there are no proper state vectors with eigenvalues (g^+, g^-), the Hilbert space $\mathscr{H}(g^+, g^-)$ is a well defined comcept. All operators in $\mathscr{C}$ have the form

$$C = \iint \delta g^+(\boldsymbol{k})\, \delta g^-(\boldsymbol{k})\, C\big(g^+(\boldsymbol{k}), g^-(\boldsymbol{k})\big) , \tag{2.17}$$

([7]) K. F. NOVOBATSKY: *Zeits. f. Phys.*, **111**, 292 (1938-39); J. G. VALATIN: *Det. Kong. Danske Vidensk. Sels. mat. fys. medd.*, **28**, No. 13 (1951).

([*]) If A is an operator with a continuous spectrum with weight function $\varrho(\lambda)$, any vector Ψ in the Hilbert space on which A is defined can be written

$$\Psi = \int \Psi(\lambda) \varrho(\lambda)\, \mathrm{d}\lambda ,$$

where $\Psi(\lambda)$ for varying λ are the components of Ψ with respect to the spectrum of A, and we can write symbolically

$$\mathscr{H} = \int \oplus \mathscr{H}(\lambda) \varrho(\lambda)\, \mathrm{d}\lambda$$

and

$$|\Psi|^2 = \int |\Psi(\lambda)|^2 \varrho(\lambda)\, \mathrm{d}\lambda ,$$

where $|\Psi(\lambda)|$ is the norm of $\Psi(\lambda)$ in the « subspace » $\mathscr{H}(\lambda)$. This is called the direct integral of the Hilbert spaces $\mathscr{H}(\lambda)$, and they need only be defined to within a set of measure zero. See references ([6]) for more details.

the generalized diagonalization of C with respect to the eigenvectors of χ^+, χ^-. Non gauge invariant operators are those which connect one $\mathscr{H}(g^+, g^-)$ with another. If Ψ is any vector in $\mathscr{H}$, the normalization can be written

$$(2.18) \qquad |\Psi|^2 = \iint \delta g^+(\boldsymbol{k})\,\delta g^-(\boldsymbol{k})\,|\Psi(g^+(\boldsymbol{k}), g^-(\boldsymbol{k}))|^2,$$

where $\Psi(g^+(\boldsymbol{k}), g^-(\boldsymbol{k}))$ is a proper state vector in $\mathscr{H}(g^+, g^-)$, such that the integral of $|\Psi(g^+, g^-)|^2$ over g^+, g^- exists. The set of such normalizable vectors will be possible physical states. The Hilbert space $\mathscr{H}(0, 0)$ has the special property of being an invariant space in the Lorentz gauge, whilst the remaining spaces transform amongst themselves according to

$$\mathscr{L}\mathscr{H}(g^+(\boldsymbol{k}), g^-(\boldsymbol{k})) \to \mathscr{H}(g^+(\mathscr{L}\boldsymbol{k}), g^-(\mathscr{L}\boldsymbol{k}))\,.$$

In the remainder ring $\mathscr{C}^L$, the vacuum state is defined by

$$(2.19) \qquad H^L\Psi_0^L = 0\,, \qquad \text{or} \qquad \mathscr{A}_\mu^+(\boldsymbol{k})\Psi_0^L = 0\,,$$

where $\mathscr{A}_\mu^+(\boldsymbol{k})$ are the positive frequency components of the transverse potentials (or an equivalent definition in terms of $F_{\mu\nu}^+(\boldsymbol{k})$ could be given). Such a state can be considered as a vector in the space $\mathscr{H}(0, 0)$. But in each of the subspaces $\mathscr{H}(g^+, g^-)$, equivalent vacuum states can be defined by

$$(2.19') \qquad H^g\Psi_0^g = 0\,,$$

and many particle states generated by successive applications of $\mathscr{A}_\mu^-(\boldsymbol{k})$. The correspondence $\mathscr{H}(0, 0) \to \mathscr{H}(g^+, g^-)$ generated by a (non-Lorentz) gauge transformation preserves the particle number relations so that the matrix elements of gauge invariant quantities in any of the spaces $\mathscr{H}(g^-, g^-)$ will be in correspondence, *i.e.*

$$(\Psi^g(N_1', N_2', \ldots)\,, \quad C^g\Psi^g(N_1, N_2, \ldots)) = (\Psi^L(N_1', N_2', \ldots)\,, \quad C^L\Psi^L(N_1, N_2, \ldots))\,,$$

where $C^g \in \mathscr{C}^g$ and $C^L \in \mathscr{C}^L$ correspond.

Then if

$$\Psi(N_1, N_2, \ldots) = \int \delta g^+(\boldsymbol{k})\,\delta g^-(\boldsymbol{k})\,\alpha(g^+(\boldsymbol{k}), g^-(\boldsymbol{k}))\Psi^g(N_1, N_2, \ldots)\,,$$

is a state of indefinite gauge, but definite transverse photon particle numbers, the normalization condition will give

$$\int \delta g^+(\boldsymbol{k})\,\delta g^-(\boldsymbol{k})\,|\alpha(g^+(\boldsymbol{k}), g^-(\boldsymbol{k}))|^2 = 1\,,$$

and the matrix elements of a gauge invariant operator between two states of different particle numbers but the *same* gauge components $\alpha(g^+(\boldsymbol{k}), g^-(\boldsymbol{k}))$ will be

$$(\Psi(N_1', N_2', \ldots), C\Psi(N_1, N_2, \ldots)) = \int \delta g^+(\boldsymbol{k})\,\delta g^-(\boldsymbol{k})\,|\alpha(g^+(\boldsymbol{k}), g^-(\boldsymbol{k}))|^2 \cdot$$

$$\cdot(\Psi^L(N_1', N_2', \ldots), C^L\Psi^L(N_1, N_2, \ldots)) = (\Psi^L(N_1', N_2', \ldots), C^L\Psi^L(N_1, N_2, \ldots)),$$

independent of the gauge.

The covariant definition of the vacuum

$$A_\mu^+(\boldsymbol{k})\Psi_0 = 0 \qquad (2.19'')$$

is not a gauge-invariant definition, so that Ψ_0 will have an indefinite gauge. But, so long as we confine ourselves to states with the same gauge components, the matrix elements of gauge invariant quantities with this definition will be the same as above, *i.e.* the covariant definition is the same as the usual definition of the vacuum so far as gauge invariant quantities are concerned. This was noted by COESTER and JAUCH (7), DYSON (8), and MA (2).

The algebraic approach is essentially implied in the work of HEISENBERG and PAULI (9) in their original papers on quantum electrodynamics, whereas Fermi's approach (1) was to use the Hilbert space $\mathscr{H}(0, 0)$.

The use of the space $\mathscr{H}$ instead of $\mathscr{H}(0, 0)$ corresponds to embedding the physical space of physical states in a larger space to permit the use of the Hamiltonian formalism. The larger space, being just a mathematical artifice to enable the simpler methods of the canonical formalism to be used, need have no real existence, and so the problem of the non-normalizability of states in $\mathscr{H}(0, 0)$ with respect to the space $\mathscr{H}$ is one without physical meaning. An analogous problem would arise if the four dimensional space time of general relativity were rejected on the ground that it represented a surface, and so had zero volume in a Euclidean space of 10 dimensions, and that only Euclidean spaces of finite volume were considered to have reality.

The form of the Lorentz transformations in the Schrödinger representation depends on the choice of the equations of motion, and hence on the choice of gauge. This contrasts with the form of the Lorentz transformation in the Heisenberg representation, which is independent of the equation of motion, but which, depending on field operators at different space time points, is only

(8) F. J. DYSON: *Phys. Rev.*, **75**, 492 (1949).

(9) W. HEISENBERG and W. PAULI: *Zeits. f. Phys.*, **56**, 1 (1929); **59**, 168 (1930); L. ROSENFELD: *Ann. d. Phys.*, **5**, 113 (1930).

completely determined when the solutions of the equations of motion are known. So, if a different gauge is chosen, the Lagrangian will be different, and the Lorentz transformations will also be different.

3. – Electromagnetic field with interaction.

For simplicity, the case of the interaction of a charged spinor field with an electromagnetic field will be considered. We consider a family of operators $q_\mu(\boldsymbol{k})$, $p_\mu(\boldsymbol{k})$, $\varphi_\sigma(\boldsymbol{k})$, $\theta_\sigma(\boldsymbol{k})$ which are restricted by the commutation relations (2.1), and

$$(3.1)\qquad \{q_\sigma(\boldsymbol{k}), \theta'_\sigma(\boldsymbol{k}')\} = i\hbar\,\delta_{\sigma\sigma'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,, \qquad \{q_\sigma(\boldsymbol{k}), q_{\sigma'}(\boldsymbol{k}')\} = \{\theta_\sigma(\boldsymbol{k}), \theta_{\sigma'}(\boldsymbol{k}')\} = 0\,,$$

and the reality conditions (2.2) and

$$(3.2)\qquad \theta_\sigma(\boldsymbol{k}) = i\hbar\,\varphi_\sigma(\boldsymbol{k})\,.$$

The algebra of operators generated by $\theta_\sigma(\boldsymbol{k})$ and $q_\sigma(\boldsymbol{k})$ will be a $*$-algebra $\mathscr{A}_M$, and the direct product of $\mathscr{A}_M$ and $\mathscr{A}_E$ will be the operator $*$-algebra $\mathscr{A} = \mathscr{A}_M \times \mathscr{A}_E$ of quantum electrodynamics. The representation space of $\mathscr{A}$ is the direct product of the representation spaces $\mathscr{H}_M$ and $\mathscr{H}_E$ of $\mathscr{A}_M$ and $\mathscr{A}_E$ respectively. On $\mathscr{A}$ is imposed a family of inner automorphisms which are a representation of the inhomogeneous Lorentz group. As in Section 2, this representation will be gauge dependent. The equations of motion (2.3) are changed to

$$(3.3)\qquad \left\{\begin{aligned}
&\dot{A}_j = c^2 \Pi_j + ic\,\frac{\partial A_4}{\partial x_j}\,, \qquad \dot{A}_4 = c^2 \Pi_4 - ic\,\frac{\partial A_j}{\partial x_j}\,,\\
&\dot{\Pi}_j = -\frac{\partial^2 A_l}{\partial x_j\,\partial x_l} + \frac{\partial^2 A_j}{\partial x_l^2} - ic\,\frac{\partial \Pi_4}{\partial x_j} + e\psi^+\gamma_j\psi\,, \qquad \dot{\Pi}_4 = ic\,\frac{\partial \Pi_j}{\partial x_j} + e\psi^+\gamma_4\psi\,,\\
&\dot{\psi}_\sigma = -ic\,\frac{\partial}{\partial x_k}(\gamma_4\gamma_k\psi)_\sigma - i\mu c(\gamma_4\psi)_\sigma + \frac{ie}{\hbar}A_4\psi_\sigma - \frac{e}{\hbar}A_k(\gamma_4\gamma_k\psi)_\sigma\,,\\
&\dot{\pi}_\sigma = -ic\,\frac{\partial}{\partial x_k}(\gamma_4\gamma_k\pi)_\sigma + i\mu c(\gamma_4\pi)_\sigma - \frac{ie}{\hbar}A_4\pi_\sigma + \frac{e}{\hbar}A_k(\gamma_4\gamma_k\pi)_\sigma\,.
\end{aligned}\right.$$

π_σ and ψ_σ are the Fourier transforms of θ_σ and φ_σ respectively, μ is the rest mass of the charged particle, and $\psi^+ = i\overline{\psi}\gamma_4$.

The supplementary conditions are now

$$\chi(x)\Psi = 0 = \dot{\chi}(x)\Psi\,,$$

where

$$\chi(x) = -\,ic\Pi_4(x)\,, \qquad \text{and} \qquad \dot{\chi}(x) = -\,ic\dot{\Pi}_4(x) = c^2\,\frac{\partial \Pi_j}{\partial x_j} - ice\psi^+\gamma_4\psi\,.$$

Combinations of (3.5) analogous to (2.9) give

$$(3.6)\qquad \left\{\begin{aligned} \chi^+(\boldsymbol{k}) &= -\,ic\left(\overline{k_\mu p_\mu(\boldsymbol{k})} + \frac{1}{c^2}\,j_4(-\,\boldsymbol{k})\right),\\ \chi^-(\boldsymbol{k}) &= -\,ic\left(k_\mu p_\mu(\boldsymbol{k}) + \frac{1}{c^2}\,j_4(-\,\boldsymbol{k})\right). \end{aligned}\right.$$

Lorentz transformations analogous to (2.10) are easily found, and the constraints determined by (3.6) can be shown to be Lorentz invariant families- $\chi^+(\boldsymbol{k})$ is also a normal operator, with $\overline{\chi^-(\boldsymbol{k})} = \chi^-(-\,\boldsymbol{k})$. $j_4(\boldsymbol{k})$ is the Fourier transform of the spinor charge density, so that

$$ec\psi^+\gamma_4\psi = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3k\,\exp\,[i\boldsymbol{k}\cdot\boldsymbol{x}]\,j_4(\boldsymbol{k})\,,$$

or, inverting,

$$(3.7)\qquad j_4(\boldsymbol{k}) = \frac{ec}{\hbar}\,\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3k'\,\theta_\sigma(\boldsymbol{k}+\boldsymbol{k}')\,\varphi_\sigma(\boldsymbol{k}')\,.$$

The reality condition is $\overline{j_4(\boldsymbol{k})} = -\,j_4(-\,\boldsymbol{k})$.

The development described in Section 2 can now be carried through with the construction of the commutative algebra $\mathscr{B}(\chi^-)$ and $\mathscr{B}(\chi^-)$, and the invariant commutator $*$-algebra $\mathscr{C}$ of gauge invariant operators. The operator T generates a Lorentz gauge transformation (2.13), together with

$$(3.8)\qquad \psi_\sigma(x,t) \to \psi_\sigma(x,t) + \frac{ie}{\hbar c}\,\Lambda(x,t)\,\psi_\sigma(x,t)\,,$$

for the Heisenberg representation. In the Schrödinger representation (3.8) is obtained for $t=0$. The algebra of electromagnetic quantities in $\mathscr{C}$ is generated by the same operators as before, as the interaction does not affect their commutation relation with χ^+ and χ^-. The spinor field operators are not gauge invariant, as (3.8) demonstrates.

The unitary transformation $a^s = Sa\bar{S}$, with

$$S = \exp\left\{-\,\frac{i}{\hbar c^2}\int \frac{j_4(-\,\boldsymbol{k})\,k_l g_l(\boldsymbol{k})}{k^2}\,\mathrm{d}^3k\right\},$$

gives

$$
(3.9)\qquad \begin{cases} q_i^s = q_i\,, \qquad q_4^s = q_4\,, \qquad p_i^s(\boldsymbol{k}) = p_i(\boldsymbol{k}) + \dfrac{k_i j_4(-\boldsymbol{k})}{c^2 k^2}\,, \\[2ex] \psi_\sigma^s(x) = \exp\left(-\dfrac{ie}{4\pi\hbar c}\displaystyle\int \dfrac{\operatorname{div}\boldsymbol{A}}{r}\,\mathrm{d}^3x'\right)\psi_\sigma(x)\ (^*)\,, \\[2ex] \pi_\sigma^s(x) = \exp\left(\dfrac{ie}{4\pi\hbar c}\displaystyle\int \dfrac{\operatorname{div}\boldsymbol{A}}{r}\,\mathrm{d}^3x'\right)\pi_\sigma(x)\,, \end{cases}
$$

(the x-representation for the spinors gives a more perspicuous form). The commutation relations (2.1) and (3.1) are unchanged by (3.9), and, in addition

$$[\chi^\pm, \psi_\sigma^s] = [\chi^\pm, \pi_\sigma^s] = 0\,,$$

so that ψ_σ^s, π_σ^s are gauge invariant spinors. Furthermore, as

$$(3.6')\qquad \chi^+(\boldsymbol{k}) = -ic\bar{k}_\mu p_\mu^s(\boldsymbol{k})\,, \qquad \chi^-(\boldsymbol{k}) = -ick_\mu p_\mu^s(\boldsymbol{k})\,,$$

the gauge invariance of the transformed spinors is manifest. In this representation (the Coulomb representation), the algebra $\mathscr{C}$ is constructed from the same combinations of the generators q_μ^s, p_μ^s as were used in the free field case, together with the spinor operators ψ_σ^s, π_σ^s, *i.e.* $\mathscr{C} = \mathscr{C}_E^s \times \mathscr{A}_M^s$. $\mathscr{C}_E^s$ and $\mathscr{A}_M^s$ are not separately Lorentz invariant. The remainder ring $\mathscr{C}^L$ in the Lorentz gauge is equal to $\mathscr{C}_E^L \times \mathscr{A}_M^s$, and so is generated by the same expressions in terms of q_μ^s and p_μ^s as were used in terms of q_μ and p_μ in the free field case *e.g.* the Novobatsky or Valatin form, along with ψ_σ^s and π_σ^s.

The evolution of the operators in this representation will be governed by the Hamiltonian with the Coulomb term explicit, and the definite metric in the representation space of $\mathscr{C}^L$ can be employed once more. Because the stationary states of the interaction Hamiltonian are not known, the definition of the vacuum and many-particle state can only be given by some approximated method, *e.g.* via the interaction representation.

4. – Alternative approaches.

The approach described in Sections **2** and **3** is in terms of the conventional theory, with the conventional supplementary condition. Modifications to this

(*) The exponent in (3.9) can be written, apart from the numerical factor, as

$$\int_{-\infty}^{x}\mathrm{d}x'\cdot\boldsymbol{A} + \int_{-\infty}^{x}\mathrm{d}x''\cdot\int\mathrm{d}^3x'\,\frac{\nabla'\times(\nabla'\times\boldsymbol{A})}{r}\,,$$

and as the second factor is gauge invariant, it can be dropped without affecting the gauge invariance of ψ^s. However the remaining term is not, in general, single valued.

approach have been described by CANDLIN [1], who introduced a new supplementary condition, and GUPTA [1], who used a different metric. The algebraical content of these variations, and the reasons why they lead to the same physical conclusions will now be discussed.

a) Candlin's approach. CANDLIN introduced a number of auxiliary functions, which in this notation can be written

$$\chi_1(x) = -\,ic\Pi_4(x)\,, \qquad \chi_2(x) = c\,\frac{\partial \Pi_j}{\partial x_j}\,,$$

$$\varphi_1(x) = (\nabla^2)^{-1}\left(i\,\frac{\partial^2 A_4}{\partial x_j^2} + \frac{1}{2}\,c\,\frac{\partial \Pi_j}{\partial x_j}\right), \qquad \varphi_2(x) = -\,(\nabla^2)^{-1}\left(\frac{\partial A_j}{\partial x_j} + \frac{1}{2}\,ic\Pi_4\right). \tag{4.1}$$

with Fourier transforms

$$\begin{cases} \chi_1(\boldsymbol{k}) = -\,icp_4(\boldsymbol{k})\,, \qquad \chi_2(\boldsymbol{k}) = ick_j p_j(\boldsymbol{k})\,, \\ \varphi_1(\boldsymbol{k}) = i\left(q_4(\boldsymbol{k}) - \dfrac{ck_j}{2k^2}\,p_j(-\,\boldsymbol{k})\right), \qquad \varphi_2(\boldsymbol{k}) = \dfrac{i}{k^2}\left(k_j q_j(\boldsymbol{k}) + \tfrac{1}{2}cp_4(-\,\boldsymbol{k})\right), \end{cases} \tag{4.2}$$

(in the free field case).

The supplementary conditions which he proposed were:

$$\begin{cases} C_1(\boldsymbol{k})\Psi = \dfrac{i}{k}\left(\chi^+(\boldsymbol{k}) - 2ik^2 q_4(-\,\boldsymbol{k})\right)\Psi = 0\,, \\ C_2(\boldsymbol{k})\Psi = \left(\chi^-(\boldsymbol{k}) - 2kk_j q_j(-\,\boldsymbol{k})\right)\Psi = 0\,. \end{cases} \tag{4.3}$$

It can be readily verified that $[C_1(\boldsymbol{k}), C_2(\boldsymbol{k})] = 0$, but $C_1(\boldsymbol{k})$, $C_2(\boldsymbol{k})$ are not normal operators. The commutator algebras $\mathscr{C}(C_1)$, $\mathscr{C}(C_2)$ can be generated in the usual way. The remainder ring $\mathscr{C}^L$ has exactly the same structure as before, so that all the physical consequences are unaltered, but now $\mathscr{H}(0, 0)$ is a subspace of $\mathscr{H}$, so that the components $\mathscr{H}(0, 0)$ can be normalized as vectors in $\mathscr{H}$ rather than in $\mathscr{H}(0, 0)$. However the operators $C_1(\boldsymbol{k})$, $C_2(\boldsymbol{k})$ do not form a Lorentz invariant family, so the commutator algebras $\mathscr{C}(C_1)$ and $\mathscr{C}(C_2)$ are not invariant sub-algebras. But the remainder ring $\mathscr{C}^L$, having the same structure as before, has the same automorphisms, and so is invariant. The approach suggested by CANDLIN could be generalized by replacing χ^+ and χ^- by any operators for which $\mathscr{C}^L$ is unchanged. The advantage of having $\mathscr{H}(0, 0)$ a subspace is only apparent when the potentials are used explicitly.

The replacement of χ^+ and χ^- by C_1 and C_2 can be represented formally by a similarity transformation. If $S = \exp\left(i\int G(k)\,\mathrm{d}^3k\right)$, where

$$G(\boldsymbol{k}) = \frac{ik}{\hbar c}\left(q_j(\boldsymbol{k})q_j(-\,\boldsymbol{k}) - q_4(\boldsymbol{k})q_4(-\,\boldsymbol{k}) - \frac{ik_j}{k}\,q_4(\boldsymbol{k})\,q_j(-\,\boldsymbol{k}) + \right.$$
$$\left. + \frac{ik_j}{k}\,q_4(\boldsymbol{k})q_j(\boldsymbol{k})\right), \tag{4.4}$$

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then

$$(4.5)\qquad \begin{cases} S\chi^+S^{-1} = \chi^+ - 2ik^2q_4(-\boldsymbol{k}) = \dfrac{i}{k}\,C_1(\boldsymbol{k})\,, \\ S\chi^-S^{-1} = \chi^- - 2kk_jq_j(-\boldsymbol{k}) = \quad C_2(\boldsymbol{k})\,, \end{cases}$$

This transformation has only formal significance because it is not well defined, being the exponential of an unbounded Hermitean operator. However it shows, formally, how the algebraical relationships of Section 2 are preserved, and can be replaced by a homomorphic mapping which is not necessarily an inner automorphism. Such a transformation (4.4) cannot be well defined because it changes the spectral properties of χ^+ and χ^-. It is interesting to note the expressions for $C_1(\boldsymbol{k})$ and $C_2(\boldsymbol{k})$ in terms of the annihilation and creation operators (2.4):

$$(4.3')\qquad \begin{cases} C_1(\boldsymbol{k}) = i\sqrt{\dfrac{2\hbar c}{k}}\,(ika_4(-\boldsymbol{k}) - k_ja_j(-\boldsymbol{k}))\,, \\ C_2(\boldsymbol{k}) = -\sqrt{2\hbar kc}\,(ika_4(-\boldsymbol{k}) + k_ja_j(-\boldsymbol{k})). \end{cases}$$

(4.3′) contains only annihilation operators.

b) Gupta's approach. The first of conditions (4.3) only is taken to be the supplementary condition, so that there are now an infinite number of normalizable states with fixed transverse photon numbers which satisfy the supplementary condition. But this ambiguity is removed by interpreting the expectation values by means of the indefinite metric. An Hermitean operator η is introduced with the properties

$$(4.6)\qquad [a_j(\boldsymbol{k}), \eta] = 0 = \{a_4(\boldsymbol{k}), \eta\}\,, \qquad\qquad \eta^2 = E.$$

If we introduce two number operators $N^{(1)}(\boldsymbol{k})$, $N^{(2)}(\boldsymbol{k})$ by the definitions

$$(4.7)\qquad N^{(1)}(\boldsymbol{k}) = \frac{k^3}{4\hbar c}\,\overline{C_1(k)}\,C_1(k)\,, \qquad N^{(2)}(k) = \frac{k}{4\hbar c}\,\overline{C_2(\boldsymbol{k})}\,C_2(\boldsymbol{k})\,,$$

then the solutions of (4.3) for a fixed $\boldsymbol{k}$ can be labelled Ψ_{00}, Ψ_{01}, Ψ_{02}, ... with the first suffix referring to the eigenvalues of $N^{(1)}(\boldsymbol{k})$, and the second to the eigenvalues of $N^{(2)}(\boldsymbol{k})$, for $N^{(2)}(\boldsymbol{k})$ commutes with $C_1(\boldsymbol{k})$. As

$$\eta N^{(2)}(\boldsymbol{k}) = \frac{1}{k^4}\,N^{(1)}(\boldsymbol{k})\eta\,,$$

we have

$$(\Psi_{0N}, \eta N^{(2)}(\boldsymbol{k})\Psi_{0N}) = N(\Psi_{0N}, \eta\Psi_{0N}) = \frac{1}{k^4}\,(\Psi_{0N}, N^{(1)}(\boldsymbol{k})\eta\psi_{0N}) = 0\,.$$

Hence $(\Psi_{0N}, \eta\Psi_{0N}) = 0$ unless $N=0$. So, with this normalization only the simultaneous eigenstate of both conditions (4.3) contributes. Also for any operator $C^L \in \mathscr{C}^L$ which commutes with both operators (4.3), we have

$$(\Psi_{0N}, C^L\Psi_{0N}) = 0 \qquad \text{for } N \neq 0.$$

But for operators which do not commute with the second operator in (4.3), there will be a variation in expectation value with N, because such an operator will have non-zero matrix elements between different eigenstates of $N^{(2)}(\boldsymbol{k})$. So although the physical consequences of the approaches of GUPTA and CANDLIN will be the same, there will be a difference in the treatment of the non gauge invariant operators because in Candlin's approach the states Ψ_{0N} are rejected.

Algebraically, the first stage of the Gupta approach of requiring $A_4(x)$ to be a Hermitean, rather than antiHermitean, operator, can be represented as the homomorphism

$$\text{(4.8)} \qquad \overline{a_4(\boldsymbol{k})} \to a_4(-\boldsymbol{k})\,, \qquad a_4(\boldsymbol{k}) \to a_4(\overline{-\boldsymbol{k}})\,,$$

which carries $q_4(\boldsymbol{k})$ into

$$q_4(\boldsymbol{k}) = \sqrt{\frac{\hbar c}{2k}}\,(a_4(-\boldsymbol{k}) + a_4(\boldsymbol{k}))\,,$$

the Fourier transform of an Hermitean operator. The mapping (4.8) carries χ^+ into χ^+_G, but $\chi^- \to -ik\,\overline{a_4(\boldsymbol{k})} - k_j a_j(\boldsymbol{k})$ which is not proportional to $C_2(\boldsymbol{k})$. But both $C_2(\boldsymbol{k})$ and χ^-_G commute with χ^+_G, and either one may be taken. $C_2(\boldsymbol{k})$ has the advantage of containing annihilation operators only, and so leads to a simpler mathematical and physical interpretation (although the necessity for a physical interpretation of longitudinal and scalar photons does not seem to be great). The mapping (4.8) leaves the commutation relations (2.5) and multiplication properties unchanged, but violates the reality properties of the operators. So the inverse of a unitary operator remains the inverse under this mapping, but is no longer the Hermitean conjugate. So, for example, the inner automorphisms induced by Lorentz transformations are still inner automorphisms, but no longer unitary transformations. Instead they are similarity transformations. In particular the evolution operators $U(t) = \exp[iHt]$ will no longer be unitary, but the transformation $A(t) = U(t)A(0)U(t)^{-1}$ will be a similarity transformation.

The gauge invariant algebra $\mathscr{C}^L$ is invariant under this mapping, and so the physical properties are preserved.

Instead of the conditions (4.3), an equivalent set of operators

$$
\left\{
\begin{aligned}
&\alpha_4 a_4(\boldsymbol{k}) + \overline{\alpha}_4 \overline{a_4(\boldsymbol{k})} + \alpha_l \frac{k_j}{k} a_j(\boldsymbol{k}) + \overline{\alpha}_l \frac{k_j}{k} \overline{a_j(\boldsymbol{k})}\,,\\
&\beta_4 a_4(\boldsymbol{k}) + \overline{\beta}_4 \overline{a_4(\boldsymbol{k})} + \beta_l \frac{k_j}{k} a_j(\boldsymbol{k}) + \overline{\beta}_l \frac{k_j}{k} a_j(\boldsymbol{k})\,,
\end{aligned}
\right. \tag{4.11}
$$

($\overline{\alpha}$, $\overline{\beta}$ not necessarily complex conjugates of α and β), could be chosen, subject to the restriction that they commute and are independent operators. The remainder ring $\mathscr{C}^L$ would be unchanged and the normalization of the states could be varied by suitable choice of the α's and β's.

RIASSUNTO (*)

Si espone la condizione supplementare richiesta dall'elettrodinamica quantistica in una forma esente da difficoltà di normalizzazione anche con metrica definita. Si tratta la condizione supplementare come un generatore di una sotto-algebra commutativa nell'anello degli operatori di campo. Questa sotto-algebra genera un ideale nella propria algebra commutativa e il corrispondente anello residuo è formato dalla serie di tutti gli operatori fisici invarianti di gauge. Qualsiasi rappresentazione di questo anello residuo è esente da difficoltà di normalizzazione. Si discute la relazione di questo procedimento con quello di Fermi, Candlin e Gupta (1).

(*) *Traduzione a cura della Redazione.*

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Cosmic Ray Events in November 1960.

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(ricevuto il 12 Aprile 1961)

Summary. — The principal cosmic ray (C.R.) events of a remarkable group of solar and geophysical phenomena observed on November 10 to 15, 1960 are reported. Particular attention is called to the C.R. increase recorded on November 12 at high latitude stations ($\geqslant 45°$ geom.) and in the nucleonic component only, to its second, well distinct peak and to certain peculiarities of the whole phenomenon. In order to explain the 5 hours delay in the access to the Earth of the solar C.R. producing this enhancement, it seems necessary to look for some « storage » mechanism and some release process thereafter.

1. – Introduction.

A remarkable series of solar and geophysical phenomena in close succession was observed during the period November 10 to 15, 1960.

Thanks to the kindness of several cosmic ray stations all over the world (see Tables I and II), we were enabled to compare our recording data with those of similar detectors at different latitudes and altitudes. The succession of the principal events is shown in Table I.

No cosmic ray increases, either on the 12th or the 15th, were detected by our neutron monitor standard pile in Rome, as it mostly happens at our geomagnetic latitude. This fact is indicative of an upper limit in the rigidity of the added cosmic ray particles arriving from the Sun to higher latitude stations. A similar implication might be drawn from the absence of any increase in the total and meson components not only in Rome, but apparently anywhere on the surface of the Earth.

As regards our recordings, the intensity level keeps approximately constant until about 19.00 UT of Nov. 12; then a sharp Forbush-type decrease takes place, apparently simultaneous with the onset of a series of strong perturbations superimposed on the main phase of a geomagnetic storm which had its SSC at 13.48 UT, *i.e.* about 5 hours earlier.

TABLE I.

Data	Optical-flares (*)				Solar-radio-response (type IV) (**)		Geomagnetic data (**)	Cosmic-rays data	
	Position	Importance	Start time (UT)	Duration (min)	Start time (UT)	Obs. frequency (MHz)		Neutrons data at high latitude	Neutrons data at low latitude (and all meson data)
Nov. 10	28E 29N	3	10.09	178	11.16	2980-545-200	—		—
Nov. 11	—	—			03.21	9400-545-200	—	—	—
Nov. 12	5W 26N	3+	13.25	357	13.27	2980-545-200	SSC 13.48 UT	1st Increase, onset ~ 14 UT	
							Main phase 18 UT		
							SSI 18.50 UT	2nd Increase, onset ~ 19 UT	1st F.D., onset ~ 19 UT
Nov. 13	—	—	—				SSI 10.22 UT	—	2nd Decrease, ~13 UT
Nov. 14	19W 29N	2	02.46	214	03.19	9400-545-200		—	—
Nov. 15	32W 30N	3+	02.07	140	02.21	9400-545-200	SSC 13.03 UT	3rd Increase onset ~ 03 UT	3rd Decrease, ~ 13 UT

(*) From « Map of the Sun », Fraunhofer Institut, Fribourg.
(**) Private Communication of Dr. A. D. FOKKER, Nera.
(**) Private Communication of Dr. F. MOLINA of Istituto Nazionale di Geofisica, Roma.

Both this magnetic storm and the Forbush decrease (F.D.) display very complex structures, which might conceal a crowding of various events. For example, one could note two minor peaks of the cosmic ray intensity, that occurred on Nov. 13, in the course of the same F.D., which perhaps are concomitant with two minima of the geomagnetic recordings: these two peaks seem to concord in UT at different stations, in the neutron and the meson com-

ponents as well. Another remarkable intensity variation was observed on Nov. 14, with a maximum at about *local* noon.

But it is only to the first phase of this series of phenomena that we are going to limit our attention for the moment, precisely to the events of Nov. 12.

2. – The events of November 12.

Allowing for a reasonable time of transit from the Sun to the Earth, for the cloud of low-energy solar gas which is presumed to have caused the SSC and the F.D. of Nov. 12, these two events could be associated with either the solar radioflare of Nov. 11 ($\sim$ 34 hours delay withrespect to the SSC) or the flare of Nov. 10 ($\sim$ 51 hours delay) also optically observed. In any case, attention is called to the fact that a plasma cloud was spreading out of the Sun and already on the way to interact with the Earth's field at the time when another, larger flare broke-up, on Nov. 12 at 13.22, in the same region of the solar disc, now rotated from 28° E to about the central meridian. This last flare was seemingly the source of the cosmic ray enhancement observed the same day on the Earth in the nucleonic component. This cosmic ray increase

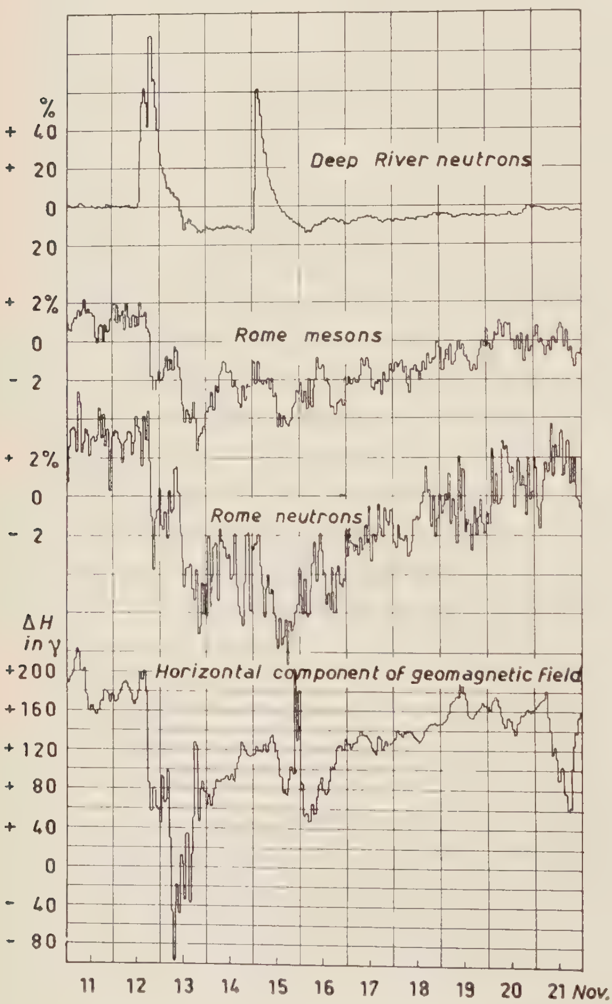

Fig. 1. – Mean hourly values of pressure corrected C.R. data in Rome and Deep River, and of horizontal component of the geomagnetic field at L'Aquila (Italy).

is actually split into two distinct peaks (see in Fig. 1 the Deep River recordings, plotted as a sample of high latitude stations). The first rise started at about 14.00 UT, *i.e.* about half an hour later than the onset of the optical flare itself, maximum cosmic ray intensity occurring around 17.00 UT; then, while the first peak was already declining toward pre-event level, a second

TABLE II.

Station ([a])	Rigidity ([b]) cut-off (GV)	Altitude (m)	09.00 I.Z. First incr.	Amplitude (%)	
				First incr. ([c])	Second incr. ([d])
Lae	14.9	s.l.	—	–	
Mina Aguilar	12.4	4000	—	—	-
Rio de Janeiro	11.5	s.l.	—	—	
Mt. Norikura	9.1	2870	—	—	
Roma	5.2	s.l.	—	—	
Hermanus	5.0	s.l.	—		
Jungfraujoch	3.8	3550	—	2	3
Zugspitze	3.3	2960	—	4	5
Monaco	3.1	500	—	3	5
Praha	2.8	228	—	4	8
Lindau	2.45	140	—	9	23
Nera	2.15	s.l.	—	13	36
London	2.1	s.l.	—	14	38
Lincoln	2.0	350	yes	22	48
Hobart	1.7	s.l.	—	40	58
Mt. Wellington	1.7	725	—	49	70
Ellsworth	1.2	s.l.	—	49	65
Upsala	1.15	s.l.	—	52	68
Ottawa	1.05	s.l.	yes	51	65
Mt. Sulphur	0.95	2280	yes	100	130
Deep River	0.85	s.l.	yes	59	71
Mawson	0.5	s.l.	—	58	73
Resolute	< 0.1	s.l.	—	62	75

(*a*) Private communication from the investigators of the various stations.
(*b*) According to Quenby-Webber.
(*c*) Mean amplitude at 16.00-18.00UT.
(*d*) Mean amplitude at 18.00-22.00UT.

larger rise started, at about 19.00 UT, *i.e.* just at the time when increased storminess was detected in the geomagnetic recordings and the cosmic ray decrease began at low latitudes in the neutron component and everywhere in the meson component. It should be recalled at this point that no further solar flare was apparently observed to be closely associated with this second cosmic ray enhancement.

Both cosmic ray increases approximately show the same features; *i.e.*, they are very broad, rather slow in rise (about 3 hours for the first increase and $1\frac{1}{2}$ hours for the second to reach peak values), and ordinary « impact zones » (I.Z., in Table II) on the Earth do not seem to be respected. All that would fit the picture of not too regular magnetic fields already present between the Sun and the Earth, which act as scattering centres for the cosmic ray particles newly emitted by the Sun.

Then, on account of the lack of perceptible anisotropies, we thought it justifiable to compare the various sea-level stations directly with one another as if their geomagnetic latitude were the only effective variable; the relative peak amplitudes were computed separately for each of the two increases of Nov. 12, and plotted in Fig. 2 *vs.* the vertical rigidity cut-offs as obtained from QUENBY and WEBBER (1). (Normalization was taken with respect to HOBART for convenience reasons.)

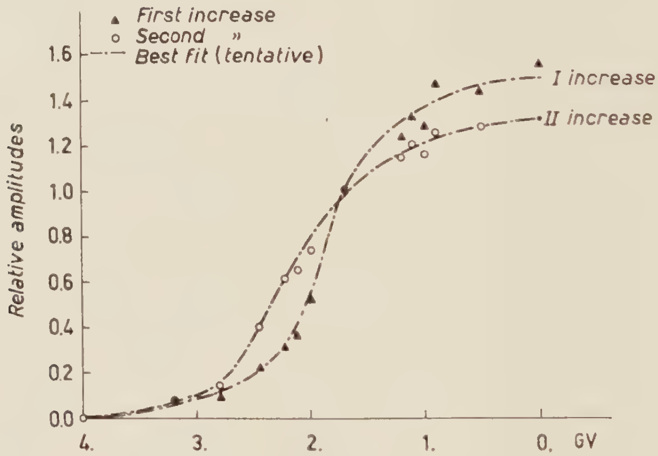

Fig. 2. – Relative amplitudes of first and second increases of Nov. 12, normalized at Hobart, versus vertical rigidity cut-offs.

When these experimental data are fitted to a power law, so as to obtain the differential rigidity spectrum of the enhanced radiation, the result is an exponent of 6 for the first increase—with a correlation coefficient 0.95—and an exponent of 4.5 for the second increase—with a correlation coefficient 0.8. On the other hand if two stations at the same latitude but different altitudes are compared (here, Deep River with Sulphur Mountain and Hobart with Mt. Wellington) an attenuation length of about 100 g/cm² is obtained for the enhanced radiation in both events of increase of Nov. 12.

(1) J. J. QUENBY and W. R. WEBBER: *Phil. Mag.*, **4**, 90 (1959).

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3. – Discussion.

The above reported circumstances seem to indicate that the second cosmic ray increase is of a difficult interpretation altogether.

To sum up, it should be recalled that its onset does not directly follow a solar flare of its own, but can only be loosely associated with the *same* flare which produced the first increase, *i.e.* with an ejection of cosmic ray particles from the Sun that started more than 5 hours earlier. What we notice, instead, is its unusual coincidence with the onset of new disturbances of the Earth's field. But no shift of the cut-off rigidities for the «galactic» cosmic rays, as one might be tempted to infer, would explain the observation of such a remarkable increase (*a*) only in the low-energy nucleonic component, differently from what is usually observed ([2]) in such cases, as, *e.g.*, for the two small peaks of Nov. 13; and *b*) at very high latitudes as well as at the intermediate ones, like, *e.g.*, at Resolute (82.9° N geom.) ([3]), where the geomagnetic field is to be considered as practically ineffective in any case.

Finally, while the attenuation coefficient obtained suggests that this added radiation is, at least on the average, just as soft as the earlier one, the exponent of its rigidity spectrum would imply, even if more inaccurately, a less steep power law; nevertheless, the enhancement is again observed neither at geomagnetic latitudes lower than 45° nor in the meson component in general.

All these aspects of the phenomenon seem to be consistent with a picture of the type that other investigators have already sketched to interpret the so-called «forbidden» protons ([4,5]).

In the present case, one beam of relativistic protons was in effect emitted with the outstanding solar flares of Nov. 12, 13.22 UT; and the first cosmic ray increase, detected on the Earth, at $\sim$14.00 UT, is a typical case of the more or less prompt arrival of solar produced cosmic ray particles with a steep rigidity spectrum. (This first event might enter in Case C of McCracken ([6]) classification.) But *only a fraction* of such particles were apparently enabled to make their almost direct route to the Earth, while the largest part got somehow *stored* within the magnetized cloud that had been previously produced and which had already spread in the interplanetary space. The storage mechanism would fail and the particles would be released right at the

([2]) I. Kondo, K. Nagashima, S. Yoshida and M. Wada: *Proceed. of Moscow C.R. Conf.*, **4**, 208 (1960).

([3]) *Geographical distribution of the I.G.Y. Stations*, (November, 1959).

([4]) T. Obayashi and Y. Hakura: *Journ. Radio Res. Lab.*, **7**, 27 (1960).

([5]) J. R. Winckler and P. D. Bhavsar: *Journ. Geoph. Res.*, **65**, 2637 (1960).

([6]) K. G. McCracken and R. A. R. Palmeira: *Journ. Geoph. Res.*, **65**, 2673 (1960).

time when *something new* seemed to occur in the interaction process between the geomagnetic field and the fields carried by the solar plasma: this would be marked by the sharp onset of strong perturbations in the geomagnetic recording, similar to a second SSC within the main phase of the first storm. For this second discharge of solar cosmic rays, the access to the Earth could then be conceived in the usual pictorial way of a spiralized descent of the particles along the newly connected lines of force of the interacting fields.

From this point of view, it is clear that a rigidity spectrum computed on the basis of the normal cut-off rigidities would be meaningless; so that no significance should be ascribed to the spectrum exponent obtained in the case of the second cosmic ray enhancement.

To confirm this picture and possibly to clarify it better, the data from all the other cosmic ray stations throughout the world and also from space vehicles will obviously be of great value.

* * *

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RIASSUNTO

Si riportano i principali eventi di Raggi Cosmici dell'insolita successione di fenomeni geofisici e solari osservati nel periodo 10-15 Novembre 1960. In particolare ci si sofferma sull'aumento registrato il 12 Novembre soltanto in stazioni a grande latitudine ($> 45°$) e nella componente nucleonica, sul suo secondo picco ben distinto e su certe peculiarità di tutto il fenomeno. Per spiegare l'accesso alla terra, con 5 ore di ritardo, dei Raggi Cosmici solari che produssero tale aumento sembrerebbe necessario supporre l'esistenza di un meccanismo di «immagazzinamento» e di un successivo processo di liberazione di tali particelle.

Recurrence Relations in Isospin for Statistical Weights.

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(ricevuto il 26 Aprile 1961)

Summary. — For statistical charge probabilities in multiple production processes, recurrence relations in isospin are derived. Taken together with an explicit formula for isoscalar charge probabilities derived in (2), these relations are the most convenient means for numerical calculations for large particle numbers and low isospin.

1. – Introduction and results.

In multiple production reactions, one may ask for the probabilities of different charge multiplicities within a given total number of produced particles. These «charge probabilities» may be calculated for systems with unique isospin under the following assumptions:

1) Charge independence.

2) Equal weights for all isospin states belonging to a given total isospin and given particle number.

These probabilities have been calculated for various cases, most extensively by CERULUS (1), who recently succeeded in getting closed formulae (2). Restricting ourselves to systems of $n = n_+ + n_0 + n_-$ pions, we adopt Cerulus' notation $^*P^T_m(n_+, n_0, n_-)$ for the charge probabilities for total isospin T and 3-component $m = n_+ - n_-$. The generalization to other systems will be evident. We

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(*) Work performed under U.S. Air Force contract nr. AF 61 (052)-47.

(1) F. CERULUS: *Suppl. Nuovo Cimento*, **15**, 402 (1960).

(2) F. CERULUS: *Nuovo Cimento*, **19**, 528 (1961).

shall prove the recurrence relation:

$$(1)\qquad {}^*P^T_m(n_+, n_0, n_-) = (C^{T,1,T-1}_{m,-1,m-1})^{-2}\left[\frac{n_-+1}{n+1}\,{}^*P^{T-1}_{m-1}(n_+, n_0, n_-+1) - \right.$$

$$\left. - (C^{T-1,1,T-1}_{m,-1,m-1})^2\,{}^*P^{T-1}_m(n_+, n_0, n_-) - (C^{T-2,1,T-1}_{m,-1,m-1})^2\,{}^*P^{T-2}_m(n_+, n_0, n_-)\right],$$

and similar relations involving ${}^*P^{T-1}_m(n_+, n_0+1, n_-)$ or ${}^*P^{T-1}_{m+1}(n_+ +1, n_0, n_-)$ instead of ${}^*P^{T-1}_{m-1}(n_+, n_0, n_-+1)$ in (1). The $C^{T_1T_2T_{12}}_{m_1m_2m_{12}}$ are vector-addition coefficients.

Some special cases have already been given by CERULUS:

$$(2.1)\qquad \begin{cases} {}^*P^1_0(n_+, n_0, n_-) = 3\,\dfrac{n_0+1}{n+1}\,{}^*P^0_0(n_+, n_0+1, n_-)\,, \\[2ex] {}^*P^1_1(n_+, n_0, n_-) = 3\,\dfrac{n_-+1}{n+1}\,{}^*P^0_0(n_+, n_0, n_-+1)\,. \end{cases}$$

Here we give recurrence relations for ${}^*P^2$ and ${}^*P^3$:

$$(2.2)\qquad \begin{cases} {}^*P^2_0 = \dfrac{5}{2}\left[\dfrac{n_0+1}{n+1}\,{}^*P^1_0(n_+, n_0+1, n_-) - {}^*P^0_0(n_+ n_0, n_-)\right], \\[2ex] {}^*P^2_1 = \dfrac{10}{3}\left[\dfrac{n_-+1}{n+1}\,{}^*P^1_0(n_+, n_0, n_-+1) - \dfrac{1}{2}\,{}^*P^1_1(n_+, n_0, n_-)\right] = \\[2ex] \qquad = 5\,\dfrac{n_-+1}{n+1}\left[\dfrac{2}{3}\,{}^*P^1_0(n_+, n_0, n_-+1) - {}^*P^0_0(n_+, n_0, n_-+1)\right], \\[2ex] {}^*P^2_2 = \dfrac{5}{3}\,\dfrac{n_-+1}{n+1}\,{}^*P^1_1(n_+, n_0, n_-+1)\,, \end{cases}$$

$$(2{,}3)\qquad \begin{cases} {}^*P^3_0 = \dfrac{7}{3}\left[\dfrac{n_0+1}{n+1}\,{}^*P^2_0(n_+, n_0+1, n_-) - \dfrac{2}{3}\,{}^*P^1_0(n_+, n_0, n_-)\right] = \\[2ex] \qquad = \dfrac{7}{3}\,\dfrac{n_0+1}{n+1}\,[{}^*P^2_0(n_+, n_0+1, n_-) - 2\,{}^*P^0_0(n_+, n_0+1, n_-)]\,, \\[2ex] {}^*P^3_1 = \dfrac{7}{2}\left[\dfrac{n_-+1}{n+1}\,{}^*P^2_0(n_+, n_0, n_-+1) - \dfrac{1}{2}\,{}^*P^2_1(n_+, n_0, n_-) - \dfrac{1}{6}\,{}^*P^1_1(n_+, n_0, n_-)\right] \\[2ex] = \dfrac{7}{2}\,\dfrac{n_-+1}{n+1}\left[{}^*P^2_0(n_+, n_0, n_-+1) - \dfrac{5}{3}\,{}^*P^1_0(n_+, n_0, n_-+1) + 2\,{}^*P^0_0(n_+, n_0, n_-+1\right. \\[2ex] {}^*P^3_2 = \dfrac{21}{10}\left[\dfrac{n_-+1}{n+1}\,{}^*P^2_1(n_+, n_0, n_-+1) - \dfrac{1}{3}\,{}^*P^2_2(n_+, n_0, n_-)\right] = \\[2ex] \qquad = \dfrac{7}{2}\,\dfrac{n_-+1}{n+1}\left[\dfrac{3}{5}\,{}^*P^2_1(n_+, n_0, n_-+1) - \dfrac{1}{3}\,{}^*P^1_1(n_+, n_0, n_-+1)\right], \\[2ex] {}^*P^3_3 = \dfrac{7}{5}\,\dfrac{n_-+1}{n+1}\,{}^*P^2_2(n_+, n_0, n_-+1)\,. \end{cases}$$

Taken together with Cerulus' formula for $^*P_0^0$:

$$^*P_0^0(n_+, n_0, n_-) = \frac{n!}{2^{2n_+} n_+! n_0! n_-!} \sum_{q=0}^{n_+} \binom{2n^+}{2q} \frac{1}{n_0 + 2q + 1}, \qquad \text{for } n \text{ even},$$

$$= \frac{n!}{2^{2n_+} n_+! n_0! n_-!} \sum_{q=0}^{n_+-1} \binom{2n_+}{2q+1} \frac{1}{n_0 + 2q + 2}, \quad \text{for } n \text{ odd},$$

formulae (2.1-3) are the most convenient means for numerical calculations for large n and low T.

2. – Proof.

The proof of the relations requires nothing but vector addition considerations. Take for example eq. (2.1). We know that any isotopic scalar function of $n+1$ pions may be written as

$$|n+1, 0, 0\rangle = \frac{1}{\sqrt{3}} \left(|n, 1, 1\rangle \pi^- + |n, 1, 0\rangle \pi^0 + |n, 1, -1\rangle \pi^+ \right).$$

The notation is $|n, T, m\rangle$. For the squares of the charge projections this means

$$P_0^0(n_+, n_0, n_- + 1) = \tfrac{1}{3} P_1^1(n_+, n_0, n_-), \quad \text{etc.} \tag{3}$$

The charge probability is given as (see ref. ([2])):

$$^*P(n_+, n_0, n_-) = \frac{n!}{n_+! n_0! n^-!} P(n_+, n_0, n_-).$$

Inserting P in (3), we get eqs. (2.1).

Generally, in order to get recurrence relations for $^*P^T(n)$ with $T > 1$, regard the $(n+1)$-pion functions with isospin $T(n+1) = T - 1$. They may be divided into three parts, characterized by the isospin of the $n \cdot \pi$ subsystem, $T(n) = T - 2$, $T - 1$, and T, respectively. Each of these parts gives rise to equations of type (3), now involving a *partial* charge projection square on its left hand side:

$$\begin{cases} {}_{T-2}P_m^{T-1}(n_+, n_0, n_- + 1) = (C_{m+1, -1, m}^{T-2, 1, T-1})^2 \cdot P_{m+1}^{T-2}(n_+, n_0, n_-), \\ {}_{T-1}P_m^{T-1}(n_+, n_0, n_- + 1) = (C_{m+1, -1, m}^{T-1, 1, T-1})^2 \cdot P_{m+1}^{T-1}(n_+, n_0, n_-), \\ {}_{T}P_m^{T-1}(n_+, n_0, n_- + 1) = (C_{m+1, -1, m}^{T, 1, T-1})^2 \cdot P_{m+1}^{T}(n_+, n_0, n_-). \end{cases} \tag{4}$$

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Since $P_m^{T-1} = {}_T P_m^{T-1} + {}_{T-1}P_m^{T-1} + {}_{T-2}P_m^{T-1}$, we can express P^T from the last row in (4) in terms of the other P's. This yields eq. (1). The same method may be applied to states containing nucleons and strange particles in addition to pions.

* * *

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RIASSUNTO (*)

Derivo delle relazioni ricorrenti nell'isospin per le probabilità statistiche di carica nei processi di produzione multipla. Queste relazioni, prese assieme con una formula esplicita per le probabilità isoscalari di carica derivate in ([2]), sono il mezzo più conveniente per i calcoli numerici per grandi numeri di particelle con basso isospin.

(*) *Traduzione a cura della Redazione.*

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On the Existence of Solutions of the Pion-Pion Dispersion Equations. - I.

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(ricevuto l'8 Maggio 1961)

Summary. — It is proved that the Chew-Mandelstam equations for pion-pion scattering with both S and P waves possess no exact solution, with the possible exception of solutions oscillating at infinity. The usual iteration procedures for solving these equations will be only asymptotically convergent in the low energy region, the convergence being worse the larger the P wave. A cut-off is therefore unavoidable. Since the expansion of the Chew-Mandelstam equations with S and P waves in powers of the coupling constant is known to be consistent in all finite orders, this is an example of a theory in which the exact solution behaves worse than any order of the perturbation expansion.

1. – Introduction.

Three years ago Mandelstam [1] set forth a program for the calculation of the amplitudes for strong interaction processes by the solution of the dispersion relations with unitarity. His original program involved the solution of equations for the spectral functions, which are functions of two variables. It was soon found [2,3], though, that this could be simplified by using the Mandelstam representation to deduce one-variable equations for the lowest partial waves. The energy at which these are expected to become inaccurate is about the same as that at which inelastic processes become important. Since we

(1) S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).
(2) G. Chew and S. Mandelstam: *Phys. Rev.*, **119**, 467 (1960).
(3) M. Cini and S. Fubini: *Ann. Phys.*, **10**, 352 (1960).

have, at present, no means of calculating these latter, it would seem that little has been lost by this further approximation.

The important question then is whether the inclusion of higher partial waves and inelastic processes would drastically affect the low energy results, or whether it would introduce only minor quantitative modifications. In the former case one would have to introduce undetermined cut-off functions to represent these effects. Now the most prominent characteristic of the experimental data on strong interaction processes is the existence of various resonances, usually in P waves. If one introduces a cut-off into the theory, one at once loses the possibility of predicting the existence and position of these, and the interest of the work is greatly reduced. Therefore it has been hoped that satisfactory results could be obtained without introducing a cut-off.

In the present work we shall investigate whether this is theoretically possible, for the special case of pion-pion scattering. We shall investigate the Chew-Mandelstam equations [2,3] for the case when both the S and P waves are allowed to have imaginary parts, and also for the case when only the S waves have imaginary parts. We shall find that only in the latter case can exact solutions without a cut-off exist. The method we shall use is to show that no consistent behaviour at infinity is possible if the P wave has a non-vanishing imaginary part. In this we have not been able to exclude solutions which oscillate at infinity, but these appear unacceptable on physical grounds.

CHEW and MANDELSTAM [4] have previously argued that their equations will need a cut-off if the P wave is resonant, but their conclusion has not been universally accepted [5,6]. The present work establishes it on a firm basis, and also goes further, for we show that a cut-off is needed if the P wave has any imaginary part at all and not merely if it is large. Even a small amount of P wave would therefore invalidate the solution, so that the S-wave-dominant solutions of CHEW, MANDELSTAM and NOYES [7] also appear not to be true ones. This is rather gratifying since all the present evidence is that they disagree strongly with experiment. Of course, there is as yet no guarantee that these S-dominant solutions would not reappear when one introduced the necessary cut-offs.

Though these results have only been proved for pion-pion scattering, it seems very plausible to suppose them to hold more generally. The conclusion then is that only for the case with S waves alone can the partial wave disper-

(4) G. CHEW and S. MANDELSTAM: *Nuovo Cimento*, **19**, 752 (1961).

(5) J. MOFFAT: *Phys. Rev.*, **121**, 926 (1961); B. BRANSDEN and J. MOFFAT: *A Numerical Determination of Coupled S and P Amplitudes for Pion-Pion Scattering*, CERN preprint (1961).

(6) J. G. TAYLOR: *The Low Energy Pion-Pion Interaction - I*, Paris, preprint (1961).

(7) G. CHEW, S. MANDELSTAM and H.P. NOYES: *Phys. Rev.*, **119**, 478 (1960).

sion relations be solved without either including the double spectral functions or representing their effects by a cut-off. One is thus back to the situation originally predicted by MANDELSTAM [1] on the basis of the unitarity conditions for the double spectral functions—only one subtraction in the Mandelstam representation is consistent, so that only the S waves can be treated apart from the rest.

In Section **2** we give the formulae we shall need. In Section **3** we prove the theorem which states that the Chew-Mandelstam equations possess no exact solutions in the case when both the S and P waves are allowed to have imaginary parts. In the case when only the S waves have imaginary parts our methods imply that any solution must go to zero at infinity. In Section **4** we examine the iteration procedures for solving the equations and conclude that when the P wave is small they will be asymptotically convergent in the low energy region, the convergence being worse the larger the P wave. Attempts to obtain large P wave solutions of the Chew-Mandelstam equations without introducing a cut-off have recently been made by BRANSDEN and MOFFAT [6] and by TAYLOR [7]. This theorem shows that their solutions are only approximate. Provided that no complex poles or ghost states are present, they would correspond to exact solutions with a cut-off.

2. – The Chew-Mandelstam equations.

We put the pion mass equal to one. The three Mandelstam invariants for pion-pion scattering are

$$S = 4(\nu + 1)\,, \qquad t = -2\nu(1 - \cos\vartheta)\,, \qquad u = -2\nu(1 + \cos\vartheta)\,, \tag{2.1}$$

where $\nu = q^2$ is the square of the centre-of-mass momentum, ϑ is the centre-of-mass scattering angle, and s, t, u are the squares of the total centre-of-mass energies for the three processes which are represented by the pion-pion Green's function.

From isotopic invariance the scattering amplitude can be expressed in terms of three functions, A, B, C, whose definitions are given by CHEW and MANDELSTAM [2]. For these we have the crossing relations

$$A(s, t, u) = A(s, u, t)\,, \qquad B(s, t, u) = C(s, u, t)\,, \tag{2.2a}$$

$$A(s, t, u) = B(t, s, u)\,, \qquad C(s, t, u) = C(t, s, u)\,, \tag{2.2b}$$

$$A(s, t, u) = C(u, t, s)\,, \qquad B(s, t, u) = B(u, t, s)\,. \tag{2.2c}$$

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The isotopic spin amplitudes for the three processes can be expressed in terms of these. Those for process I are given by

$$\text{(2.3)} \qquad \left\{\begin{aligned} A^0 &= 3A + B + C\,, \\ A^1 &= B - C\,, \\ A^2 &= B + C\,. \end{aligned}\right.$$

(2.2a) implies that A^0 and A^2 can only have even angular momentum states while A^1 can only have odd. (2.2c) with (1) reduces to

$$\text{(2.4)} \qquad \left\{\begin{aligned} & A(\nu, \cos\vartheta) = C(\nu', \cos\vartheta')\,, \\ & B(\nu, \cos\vartheta) = B(\nu', \cos\vartheta')\,, \\ & \nu' = \frac{\nu}{2}(1-\cos\vartheta) - (\nu+1)\,, \\ & \cos\vartheta' = \frac{(\nu+1)+(\nu/2)(1-\cos\vartheta)}{(\nu+1)-(\nu/2)(1-\cos\vartheta)}\,. \end{aligned}\right.$$

(2.2b) is a consequence of (2.2a) and (2.2c).

Mandelstam relations may be written down for the three amplitudes A, B, C, and are given by CHEW and MANDELSTAM [2]. The Chew-Mandelstam equations are obtained from these by assuming that only the S and P waves are large enough to have significant imaginary parts, as has been explained very well by CINI and FUBINI [3]. The real parts of all the higher partial waves are kept however. When this approximation is made the Mandelstam relations reduce to the one variable dispersion relation of Cini and Fubini:

$$\text{(2.5)} \qquad \left\{\begin{aligned} A^0(\nu, \cos\vartheta) &= -5\lambda + r_0(s) + \tfrac{1}{3}[r_0(t) + r_0(u)] + \\ &\qquad + [r_1(t)\{s-u\} + r_1(u)\{s-t\}] + \tfrac{5}{3}[r_2(t) + r_2(u)]\,, \\ A^1(\nu, \cos\vartheta) &= r_1(s)\{t-u\} + \tfrac{1}{3}[r_0(t) - r_0(u)] + \\ &\qquad + \tfrac{1}{2}[r_1(t)\{s-u\} - r_1(u)\{s-t\}] - \tfrac{5}{6}[r_2(t) - r_2(u)]\,, \\ A^2(\nu\ \cos\vartheta) &= -2\lambda + r_2(s) + \tfrac{1}{3}[r_0(t) + r_0(u)] - \\ &\qquad - \tfrac{1}{2}[r_1(t)\{s-u\} + r_1(u)\{s-t\}] + \tfrac{1}{6}[r_2(t) + r_2(u)]\,, \end{aligned}\right.$$

where

$$(2.6)\qquad \left|\begin{aligned} r_{0,2}(s) &= \frac{s-\frac{4}{3}}{\pi}\int_0^\infty \frac{\mathrm{d}\nu'\,\mathrm{Im}\,A_0^{0,2}(\nu')}{(\nu'+\frac{2}{3})(4\nu'+4-s)}\,, \\ r_1(s) &= \frac{3}{\pi}\int_0^\infty \frac{\mathrm{d}\nu'\,\mathrm{Im}\,A_1^1(\nu')}{\nu'(4\nu'+4-s)}\,. \end{aligned}\right.$$

(A_l^I is the partial wave with isotopic spin I and angular momentum l). λ is the pion-pion coupling constant.

The Chew-Mandelstam equations are then obtained by substituting (1) into (5) and (6), and extracting the S and P waves. The most important part of these equations is the expression for the imaginary parts on the left-hand cut in terms of the imaginary parts on the right-hand cut

$$(2.7)\qquad \mathrm{Im}\,A_l^I(\nu) = \frac{1}{\nu}\int_0^{-\nu-1}\mathrm{d}\nu'\,P_l\left(1+2\,\frac{\nu'+1}{\nu}\right)\cdot \\ \cdot\left\{\alpha_{I0}\,\mathrm{Im}\,A_0^0(\nu') + \alpha_{I2}\,\mathrm{Im}\,A_0^2(\nu') + 3\left(1+2\,\frac{\nu+1}{\nu'}\right)\alpha_{I1}\,\mathrm{Im}\,A_1^1(\nu')\right\},$$

for $\nu < -1$. Here

$$(2.8)\qquad \alpha_{II'} = \begin{pmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & -\frac{5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix}.$$

Together with unitarity and the partial wave dispersion relations these provide a complete set of equations for the determination of the scattering amplitudes.

These equations were solved numerically by CHEW, MANDELSTAM and NOYES (7). However, all their solutions were characterized by small P waves, which is in disagreement with all the present experimental evidence.

3. Existence of solutions of the Chew-Mandelstam equations.

In view of the numerical work just mentioned one might suppose that this problem had been solved already. We shall see, however, that this is not so, and that, if the imaginary part of the P-wave is included, their solutions are only approximate ones. Our main result is expressed by the following theorem.

Theorem 1. The Chew-Mandelstam equations possess no exact solution, with the possible exception of solutions oscillating at infinity, unless one takes the imaginary part of the P wave to vanish completely.

Proof: First we calculate the imaginary parts at $\nu = -\infty$. We get, from (2.7), for the S waves

$$\text{(3.1)}\qquad \operatorname{Im} A_0^I(-\infty) = -\lim_{\nu\to\infty}\frac{1}{\nu}\int_0^{\nu}\mathrm{d}\nu'\left\{\alpha_{I0}\operatorname{Im} A_0^0(\nu') + \right.$$
$$\left. + \alpha_{I2}\operatorname{Im} A_0^2(\nu') + \left(3+\frac{2}{\nu'}\right)\alpha_{I1}\operatorname{Im} A_1^1(\nu')\right\} + 6\alpha_{I1}\int_0^{\infty}\frac{\mathrm{d}\nu'\operatorname{Im} A_1^1(\nu')}{\nu'}\,.$$

and for the P wave

$$\text{(3.2)}\qquad \operatorname{Im} A_1^1(-\infty) = \lim_{\nu\to\infty}\frac{2}{\nu^2}\int_0^{\nu}\mathrm{d}\nu'(\nu'+1)\{\alpha_{10}\operatorname{Im} A_0^0(\nu') + \alpha_{12}\operatorname{Im} A_0^2(\nu')\} -$$
$$-\lim_{\lambda\to\infty}\frac{1}{\nu}\int_0^{\nu}\mathrm{d}\nu'\{\alpha_{10}\operatorname{Im} A_0^0(\nu') + \alpha_{12}\operatorname{Im} A_0^2(\nu')\} +$$
$$+ 6\alpha_{11}\lim_{\nu\to\infty}\frac{1}{\nu^2}\int_0^{\nu}\mathrm{d}\nu'\left\{\nu'+3+\frac{2}{\nu'}\right\}\operatorname{Im} A_1^1(\nu') -$$
$$-3\alpha_{11}\lim_{\nu\to\infty}\frac{1}{\nu}\int_0^{\nu}\mathrm{d}\nu'\left\{5+\frac{6}{\nu'}\right\}\operatorname{Im} A_1^1(\nu') + 6\alpha_{11}\int_0^{\infty}\frac{\mathrm{d}\nu'\operatorname{Im} A_1^1(\nu')}{\nu'}\,.$$

Since $\operatorname{Im} A_0^I(\nu)$ and $\operatorname{Im} A_1^1(\nu)$ go to zero like $\nu^{\frac{1}{2}}$ and $\nu^{\frac{3}{2}}$, respectively, as $\nu\to 0$, we can replace the lower limits in (1) and (2) by some $\varepsilon > 0$ in all the terms except the last one in each equation.

We now evaluate (1) and (2) by the Cesàro limits [8]

$$\text{(3.3)}\qquad g_{(C,n)}(\infty) = g(\varepsilon) + \lim_{\nu\to\infty}\int_{\varepsilon}^{\nu}\left(1-\frac{x}{\nu}\right)^n\cdot\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x\,.$$

In the one-meson approximation the only branch points of $A_l^I(\nu)$ on the right hand cut are at 0 and ∞. Thus $\operatorname{Im} A_l^I(\nu)$ are differentiable for $\nu > \varepsilon$ and (3)

[8] E. TITCHMARSH: *Theory of Fourier Integrals* (Oxford, 1948), pp. 26-27.

can be applied. By partial integration we easily find

$$(3.4)\qquad \begin{cases} g(\infty) = g_{(c,0)}(\infty)\,, \\[2ex] \lim\limits_{\nu\to\infty} \dfrac{1}{\nu}\displaystyle\int_{\varepsilon}^{\nu} \mathrm{d}\nu' g(\nu') = g_{(c,1)}(\infty)\,, \\[2ex] \lim\limits_{\nu\to\infty} \dfrac{1}{\nu^2}\displaystyle\int_{\varepsilon}^{\nu} \mathrm{d}\nu' \nu' g(\nu') = g_{(c,1)}(\infty) - \dfrac{1}{2} g_{(c,2)}(\infty)\,. \end{cases}$$

Now a standard theorem [8] states that if $g_{(c,n)}(\infty)$ exists then $g_{(c,m)}(\infty)$ exist for all $m \geqslant n$ and are equal to it.

By unitarity $\operatorname{Im} A_l^I(\nu)$ must be bounded at infinity. We now introduce the physical assumption that they tend to constants at infinity and do not oscillate. Eq. (4) and the theorem just quoted then give

$$(3.5)\qquad \operatorname{Im} A_l^I(\infty) = \lim_{\nu\to\infty} \frac{1}{\nu}\int_0^{\nu} \mathrm{d}\nu' \operatorname{Im} A_l^I(\nu') = \lim_{\nu\to\infty} \frac{2}{\nu^2}\int_0^{\nu} \mathrm{d}\nu'\,\nu' \operatorname{Im} A_l^I(\nu')\,,$$

while all the other limits in (1) and (2) vanish. We thus get

$$(3.6)\qquad \operatorname{Im} A_0^I(-\infty) = -\alpha_{I0} \operatorname{Im} A_0^0(\infty) - \alpha_{I2} \operatorname{Im} A_0^2(\infty) - 3\alpha_{I1} \operatorname{Im} A_1^1(\infty) + \alpha_{I1} G\,,$$

$$(3.7)\qquad \operatorname{Im} A_1^1(-\infty) = -12 \operatorname{Im} A_1^1(\infty) + G\,,$$

where

$$(3.8)\qquad G = 6\int_0^{\infty} \frac{\mathrm{d}\nu' \operatorname{Im} A_1^1(\nu')}{\nu'}\,.$$

These show, among other things, that $\operatorname{Im} A_l^I(-\infty)$ may go to a constant or to infinity but will not oscillate provided $\operatorname{Im} A_l^I(+\infty)$ does not. It also follows from unitarity and (2.7) that $\operatorname{Im} A_l^I(\nu)$ is bounded except possibly at $\nu = -\infty$. It is then easy to see, from the partial wave dispersion relations, that the requirement that $\operatorname{Re} A_l^I(+\infty)$ be finite implies that

$$(2.9)\qquad \operatorname{Im} A_l^I(-\infty) = \operatorname{Im} A_l^I(+\infty)\,.$$

(This is, of course, closely related to the Pomerančuk theorem [9], being an analogue of it for the individual partial waves).

[9] D. Amati, M. Fierz and V. Glaser: *Phys. Rev. Lett.*, 4, 89 (1960).

From (6), (7) and (9) G must be finite. Since

$$\mathrm{Im}\, A_1^1(\nu) \geqslant 0 \qquad \text{for } \nu \geqslant 0 \,, \tag{3.10}$$

and is assumed not to oscillate as $\nu \to \infty$, this will not be the case unless

$$\mathrm{Im}\, A_1^1(\infty) = 0 \,. \tag{3.11}$$

(6), (9) and (11) then give

$$\left\{\begin{aligned} \mathrm{Im}\, A_0^0(\infty) + 2\,\mathrm{Im}\, A_0^2(\infty) &= 6G/5 \,, \\ \mathrm{Im}\, A_0^0(\infty) + 2\,\mathrm{Im}\, A_0^2(\infty) &= -3G/2 \,, \end{aligned}\right. \tag{3.12}$$

which are obviously consistent only if $G = 0$. Eq. (7), (9) and (11) also give directly $G = 0$. It follows from (8) and (10) that $G = 0$ if and only if if $\mathrm{Im}\, A_1^1(\nu) = 0$ for all $\nu \geqslant 0$. Q.E.D.

Corollary. If the imaginary part of the P wave vanishes completely, then the S waves must go to zero at infinity if they do not oscillate.

Proof: This follows immediately from eqs. (12) with $G = 0$ and from the fact that $\mathrm{Im}\, A_0^I(\infty) \geqslant 0$. Q.E.D.

This shows that any solution of the Chew-Mandelstam equations with only the S waves having an imaginary part must go to zero at infinity. The numerical solutions of CHEW, MANDELSTAM and NOYES [7] have this behaviour in the attractive case with $\lambda < 0$. However, it is known [10] that their solutions in the repulsive case are not true solutions, even if the imaginary part of the P wave is taken to vanish, as they always contain a ghost state.

4. – Convergence of approximation methods.

All methods of solving the Chew-Mandelstam equations are based on the use of the inverse amplitudes and some iteration procedure. According to the above theorem such procedures cannot be convergent, as there is no solution for them to converge to. However, we can see that if the P wave is small they will be asymptotically convergent in the low energy region. Suppose, for example, that one is starting from a first approximation for which $\mathrm{Im}\, A_l^I(\infty) = 0$. This is the case for the solution without crossing term, and the final iterated solutions found by CHEW, MANDELSTAN and NOYES also have this behaviour in the case with λ negative. We then find from (2.7) for

[10] Private communication from S. MANDELSTAM.

the next iteration

$$\mathrm{Im}\, A_l^I(-\infty) = \alpha_{I1} G\,, \tag{4.1}$$

where G is defined by (3.8). When inserted into the dispersion relations this will contribute a term

$$(\alpha_{I1} G/\pi)\ln \nu \tag{4.2}$$

to $\mathrm{Re}\, A_l^I(\nu)$, thus violating unitarity at high energies. However the energies at which unitarity is violated will be

$$\nu \gtrsim \exp\,[1/G]\,. \tag{4.3}$$

For the S-dominant solutions the imaginary part of the P wave is very small, so that G will be very small and the energies in (3) will be very high indeed. Now in the numerical calculation the mesh-points are chosen by transforming to the variable $y = 1/(\nu+1)$ and dividing the range of either y or $y^{\frac{1}{2}}$ into equal intervals. It is then obvious that the distribution of mesh-points will be heavily biassed in favour of the low energy region. Thus, if the P wave is sufficiently small, the only mesh-point in the range (3) will be the point at infinity. At this point the inverse amplitude, which is what is being calculated, diverges logarithmically in the first approximation, so that any failure of the iteration procedure can only take place through the cancellation of two divergent quantities. This would not be observed in a numerical calculation unless special precautions were taken to do so. Thus, for small enough P wave, the fact that the iteration procedure is only asymptotically convergent would not be at all apparent in a numerical calculation. This explains why CHEW, MANDELSTAM and NOYES were able to get their S-dominant «solutions». For larger P wave one would expect the non-convergence to show up in numerical work also, however.

The logical method to eliminate these difficulties within the Chew-Mandelstam scheme would be to introduce cut-offs in the left-hand cut, representing the effect of the higher partial waves. Another approach would be simply to stop the iteration procedure as soon as reasonable convergence was obtained in the low energy region, and hope that the resulting approximate solution would be equivalent to an exact solution with a cut-off. This seems to be essentially what has been done by BRANSDEN and MOFFAT [5] and TAYLOR [6]. The method by which their solutions were constructed ensures that unitarity is satisfied and that the cuts are in the correct positions at every stage of the procedure. Therefore they must be approximate for one of two reasons: a) The presence of complex poles or ghost states. b) The failure of

the crossing conditions (2.7) in the high energy region. In this latter case their approximate solutions would correspond to exact solutions with a cut-off.

These authors introduce two extra subtraction constants into the P wave. Now the significance of these can easily be understood in the light of the theorem above. All the solutions obtained by their method will be only approximate ones. Therefore by introducing extra parameters they provide themselves with a wider choice of approximate solutions from which to choose.

The method by which BRANSDEN, MOFFAT and TAYLOR propose to select the best « solution » is by satisfying the first derivative of the crossing eqs. (2.4) at the symmetry point $\nu = -\frac{2}{3}$, $\cos\vartheta = 0$. Now one can easily see from the Cini-Fubini eqs. (2.5) that any exact solution of the Chew-Mandelstam equations would have to satisfy (2.4), and therefore also its first derivative, identically. Thus the fact that these authors have to impose this as an additional condition, in itself indicates that their solutions are not exact.

5. – Conclusions.

It was shown by MANDELSTAM [1] that in perturbation theory there cannot be more than one subtraction in the Mandelstam representation. This means that only the S waves can be treated separately, and is closely related to the non-renormalizability of theories with vector mesons and other higher spin particles. It probably also implies that only the S waves are allowed to be resonant. However, the graphs which prevent one from treating the P wave separately do not occur in the one meson approximation (in which the contribution of inelastic processes to the absorptive part on the right hand cut are neglected) [10]. Thus the Chew-Mandelstam equations, in which both the S and P waves have imaginary parts, are consistent in all orders of perturbation theory, even if the P wave is large.

Our Theorem 1 states that nevertheless they have no exact solution. The mechanism of this is that the perturbation series contains increasing numbers of logarithms of ν. These are consistent with a single subtraction in all finite orders, but when the perturbation series is summed they change the asymptotic behaviour so that a single subtraction is no longer consistent.

The present work thus provides a counterexample to the hypothesis that the asymptotic behaviour of the exact solution is the same as that of the perturbation series.

An alternative set of equations for pion-pion scattering has been proposed by SHIRKOV and others [11]. Application of the methods of Theorem 1 to

[11] A. EFREMOV, M.G. MESHCHERYAKOV, D.V. SHIRKOV and TZU: *Proceedings of the 1960 Rochester Conference on High Energy Physics*, p. 278.

these merely indicates that any exact solution must go to zero at infinity, as in the case of the Chew-Mandelstam equations with S waves only. More powerful techniques have been developed for investigating the existence of solutions in these two cases, and it is hoped to describe these in subsequent work. It is also hoped to investigate how cut-offs should be introduced into the various equations and how many will be needed. The relationship of Theorem 1 to the perturbation series and to renormalization is another subject which it is intended to investigate in more detail later.

* * *

I am grateful to Prof. S. MANDELSTAM for some very useful discussions, and also to Prof. A. SALAM and Dr. P. MATTHEWS for their interest and encouragement.

RIASSUNTO (*)

Si dimostra che le equazioni di Chew-Mandelstam per lo scattering pione-pione con onde S e P non hanno soluzione esatta, salvo la possibile eccezione di soluzioni oscillanti all'infinito. I soliti procedimenti di iterazione per la soluzione di queste equazioni saranno solo asintoticamente convergenti nella regione di bassa energia, convergenza che diventa peggiore quanto maggiore diventa l'onda P. Perciò è inevitabile un cut-off. Poichè è noto che lo sviluppo in potenze della costante di accoppiamento delle equazioni di Chew-Mandelstam con onde S e P è coerente in tutti gli ordini finiti, queste equazioni diventano l'esempio di una teoria in cui la soluzione esatta si comporta peggio di ogni ordine dello sviluppo in perturbazioni.

(*) *Traduzione a cura della Redazione.*

On the Born-Lertes Rotational Effect.

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(ricevuto il 12 Maggio 1961)

Summary. — A unified theoretical treatment of the Born-Lertes rotational effect is presented for the special case of a system of two concentric cylinders. It is shown that both the Born and the Lertes effects arise naturally from the concepts of dielectric loss and effective conductivity. A typical theoretical curve is given for the variation of the torque with frequency.

1. – Introduction.

The question of the torque exerted on a dielectric body which is either fixed in a rotating electric field or rotating in a fixed electric field has been the subject of a number of investigations since its inception in the late nineteenth century. While, depending upon the frequency of the applied field and upon the properties of the dielectric body, a combination of effects could result in a torque being produced, the mechanisms underlying the effect have normally been considered in four groups:

i) Torques based upon the anisotropy of the body.

ii) Torques based upon the conductivity of the body.

iii) Torques based upon the dielectric loss of the body.

iv) Torques based upon electrostatic repulsion.

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The mechanism i) arises from the fact that under certain circumstances the anisotropy of the body gives rise to non-parallel polarization and electric field vectors, a circumstance which can be shown to result in the production of a torque. This effect is commonly investigated at optical frequencies, and, since the work of BETH ([1]), has been regarded as well understood. It will not be discussed furher in this paper.

The mechanism ii) arises from the fact that if a body and an electric field are in relative rotational motion with respect to one another, there will be a displacement of charge over the surface of the body, and this charge will be acted upon by the field with the result that a torque is produced. The effect when the body is fixed and the field rotated was first observed by ARNO ([2]) in 1892, and that when the body is rotated by HERTZ ([3]) in 1881. The lead of HERTZ in considering the torque on a rotating body was followed subsequently by VON SCHWEIDLER ([4]) and QUINCKE ([5]). The lead of ARNO in considering a fixed body and a rotating field was followed by a number of investigators at that time, among them VON LANG ([6]), THRELFALL ([7]), and LAMPA ([8]), and has continued to arouse interest into the present decade ([9,10]). These effects were given their first satisfactory theoretical treatment by LAMPA ([8]) in 1906; his theory was qualitatively verified by LERTES ([11]) in 1921. Excellent reviews of the early literature have been given by GRAETZ ([12]) and VON SCHWEIDLER ([13]).

The mechanism iii) is commonly considered in polar fluids and arises from the finite time required for the molecular dipoles in the fluid to line up with the applied field and from the energy loss associated with this process. It was postulated on theoretical grounds by BORN ([14]) in 1920. Born's theory was

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([1]) R. A. BETH: *Phys. Rev.*, **50**, 115 (1936).
([2]) R. ARNO: *Atti Reale Accad. dei Lincei, Rendiconti*, **1** (2), 284 (1892).
([3]) H. HERTZ: *Ann. Phys.*, **13**, 266 (1881).
([4]) E. R. VON SCHWEIDLER: *Sitzber. Kgl. Akad. Wiss. (Wien), Math.-Naturwiss. Kl.*, **106** (2*a*), 526 (1897).
([5]) G. QUINCKE: *Ann. Phys.*, **59**, 417 (1896).
([6]) V. VON LANG: *Sitzber. Kgl. Akad. Wiss. (Wien), Math.-Naturwiss. Kl.*, **115** (2*a*) 211 (1906).
([7]) R. THRELFALL: *Phys. Rev.*, **4**, 457 (1896).
([8]) A. LAMPA: *Sitzber. Kgl. Akad. Wiss. (Wien), Math.-Naturwiss. Kl.*, **116** (2*a*) 1659 (1906).
([9]) E. GROSSETTI: *Nuovo Cimento*, **10**, 193 (1958).
([10]) A. CARRELLI and M. MARINARO: *Nuovo Cimento*, **11**, 262 (1959).
([11]) P. LERTES: *Zeits. Phys.*, **4**, 315 (1921).
([12]) L. GRAETZ in *Handb. d. Phys.*, Band **4** (Leipzig, 1905).
([13]) E. R. VON SCHWEIDLER in *Handbuch der Elektrizität und des Magnetismus* Band **1** (Leipzig, 1918).
([14]) M. BORN: *Zeits. Phys.*, **1**, 221 (1920).

given qualitative verification in 1921 by the work of LERTES ([15]). Following a suggestion of BREIT ([16]), DEBYE ([17]) utilized time harmonic methods and modified Born's treatment to relate the dipole torque more directly to the dielectric loss. This torque produced on fluid dipoles by a rotating electric field has been the subject of several recent experimental papers by GROSSETTI ([18-20]).

The mechanism iv) was discovered by QUINCKE ([5]) and was subsequently explained by HEYDWEILLER ([21]) as follows. A dielectric body hung between the plates of a capacitor, which has been filled with a slightly conducting dielectric liquid, will acquire a free charge distribution over its surface, the charge near the positive plate tending to be positive and that near the negative plate negative; this naturally results in the electrostatic repulsion and subsequent rotation of the body. An excellent summary of the early literature has been given by GRAETZ ([12]). The effect was independently discovered by RICHARDSON ([22]), investigated further by one of his pupils ([23]), and later given a supremely lucid explanation by VEDY ([24]). Recently, the effect has been rediscovered, reinvestigated, and re-explained by SUMOTO ([25,26]). In connection with the more scholarly investigations of this mechanism, it is of interest to note that electrostatic generators which involve rotating dielectric disks (*e.g.*, a Wimshurst machine) can, with a very slight modification, be operated as motors by applying a suitable voltage to their output terminals and allowing the disks to rotate; motors alleged to generate up to $\frac{1}{8}$ horsepower have been so constructed ([27]). The mechanism iv) will not be discussed further in this paper.

That the effect arising from mechanisms ii) and iii) will overlap in the average dielectric fluid has been recognized by all investigators since BORN. It seems, however, that no one has attempted to combine the two into a single formula. Indeed, the recent work has been exclusively experimental, the original theoretical results of LAMPA and BORN being utilized. The more recent

(15) P. LERTES: *Zeits. Phys.*, **6**, 56 (1921).
(16) G. BREIT: *Zeits. Phys.*, **11**, 129 (1922).
(17) P. DEBYE in *Handbuch der Radiologie*, Band **1** (Leipzig, 1925).
(18) E. GROSSETTI: *Nuovo Cimento*, **13**, 350 (1959).
(19) E. GROSSETTI: *Nuovo Cimento*, **18**, 21 (1960).
(20) E. GROSSETTI: *Nuovo Cimento*, **19**, 1 (1961).
(21) A. HEYDWEILLER: *Verh. Phys. Gesellschaft (Berlin)*, **16**, 32 (1897).
(22) S. W. RICHARDSON: *Nature*, **119**, 238 (1927).
(23) L. G. VEDY: *Proc Phys. Soc. (London)*, **39**, 169 (1927).
(24) L. G. VEDY: *Proc. Camb. Phil. Soc.*, **27**, 91 (1931).
(25) I. SUMOTO: *Journ. Phys. Soc. Japan*, **10**, 494 (1955).
(26) I. SUMOTO: *Repts. Sci. Res. Inst. Japan*, **32**, 41 (1956).
(27) V. E. JOHNSON: *Modern High Speed Influence Machines* (London, 1921).

discoveries in electromagnetism and the theory of dielectrics have not been applied to this problem. The principal aim of this paper is to show how, with harmonic time dependence assumed for the applied field, both effects can arise naturally from the concepts of dielectric loss and effective conductivity. It will be demonstrated that the torques due to mechanisms ii) and iii) are not strictly additive and that their separability depends strongly on the physical properties of the material being considered.

In the body of this paper will be considered the torque on a cylinder due to a known time-varying electric potential distributed over the circumference of a somewhat larger cylinder which is concentric with the first. The cylindrical geometry was first used by ARNO (2) and has been adopted here because of its relative mathematical simplicity and because of its use in a number of recent experiments (9,10,18-20). The region between the two cylinders will be assumed to be free space so that the effects of double layer formation, conceivably significant if this volume were filled with a liquid, do not exist. This postulation of free space between the cylinders possesses the added advantages *a*) of ruling out the mechanism iv), since this is known (24) to depend upon conduction through the region between the voltage source and the dielectric body, and *b*) of eliminating the effects, known to be important (7), of the contamination of the sample surface by water vapor. Further, although the inner cylinder may be a liquid contained in a thin cylindrical shell (9,10,18-20), the effect of this shell will be neglected since it has been shown to be negligible by both LERTES (11) and GROSSETTI (9).

2. – Statement and formal solution of the problem.

It is well known that, once the driving and boundary conditions have been specified, the first step in achieving a solution for the electromagnetic field vectors, or, equivalently, the electromagnetic potentials, would normally be the separation of the spatial and temporal co-ordinates. In the general case this separation can be effected through the application of integral transform techniques. This method is invaluable for problems in which the fields are not monochromatic. However, in the special case where they are, a considerable simplification is realized by the use of time-harmonic techniques. The details of this latter method are well discussed by KING (28), whose notational scheme will be followed here, and consist in essence of assuming $\exp[j\omega t]$ as the time dependence for all variables and hence writing the several equations for the electromagnetic field with the substitution $\partial/\partial t = j\omega$; the time depen-

(28) R. W. P. KING: *Electromagnetic Engineering*, vol. 1 (New York, 1945).

dence of some variable X can be recovered at will through use of the rule

$$X(t) = RE \exp[j\omega t]\, X(\omega)\,. \tag{1}$$

The use of the time-harmonic method, which is to be utilized in this paper, makes the several field quantities, the essential densities of charge and current, and especially the permittivity, permeability, and conductivity complex. In particular:

$$\varepsilon = \varepsilon' - j\varepsilon'\,, \tag{2a}$$

$$\mu = \mu' - j\mu''\,, \tag{2b}$$

$$\sigma = \sigma' - j\sigma''\,, \tag{2c}$$

The imaginary parts of these three parameters represent, respectively, time lags in polarization (dielectric loss), magnetization (magnetic loss), and convection current.

For dielectrics σ'' is ordinarily of no consequence and will not be considered here. Under the restrictions to be placed upon the applied frequency $f = \omega/2\pi$ the phase angle of the complex permeability will become unimportant. The nature of ε' and ε'' is, however, of greatest import: it is discussed *in extenso* in all modern treatises on dielectric behavior. For fluids ε' and ε'' arise, respectively, from the ordinary polarization of the medium and from the time lag of the molecular dipoles in lining up with the applied field. The elementary theory of these quantities is relatively simple and leads [29] to the expressions

$$\varepsilon' = (\varepsilon_s - \varepsilon_\infty)\,\frac{1}{1+\omega^2\tau^2} + \varepsilon_\infty\,, \tag{3a}$$

$$\varepsilon'' = (\varepsilon_s - \varepsilon_\infty)\,\frac{\omega\tau}{1+\omega^2\tau^2}\,, \tag{3b}$$

where ε_s is the permittivity measured at dc or audio frequency ac and ε_∞ is the dc permittivity as calculated from Cauchy's dispersion relation. τ is the so-called relaxation time, a measure of the time required for the molecular dipoles to align themselves with the applied field; it is commonly determined from measurements of ε''. The agreement of the eq. (3) with experiment is entirely adequate for the purposes of this paper.

The infinite coaxial geometry, with which it is proposed to deal in this

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[29] C. J. F. Böttcher: *Theory of Electric Polarization* (Amsterdam, 1952).

paper, is shown in Fig. 1. The region $r < a$ is a fluid with the constitutive parameters indicated. The region $a < r < b$ is free space. It is assumed that the region $r \leqslant a$ is electrically neutral overall. It is further assumed that $U(\theta; \omega)$, the scalar electric potential on the boundary $r = b$ admits of expansion in a Fourier series:

$$U(\theta; \omega) = \sum_{n=0}^{\infty} \{C_n(\omega) \cos n\theta + S_n(\omega) \sin n\theta\}, \tag{4}$$

$$\left.\begin{matrix} C_n \\ S_n \end{matrix}\right\} = \frac{1}{\pi} \int_0^{2\pi} U(\theta; \omega) \left\{\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\} d\theta, \qquad n \geqslant 1, \tag{5a}$$

$$\left.\begin{matrix} C_n \\ S_n \end{matrix}\right\} = \frac{1}{2\pi} \int_0^{2\pi} U(\theta; \omega) \left\{\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\} d\theta. \qquad n = 0. \tag{5b}$$

The object of this section is to derive an expression for the torque on the center rod in terms of its constitutive parameters and the above specified driving conditions.

Unfortunately, the problem as stated above is not in general soluble due to the fact that the driving conditions are stated in terms of the scalar potential φ while the vector potential $\boldsymbol{A}$ is left unspecified. To overcome this difficulty, and to effect a considerable mathematical simplification, it is convenient to assume that $\boldsymbol{A}$ can be neglected. Since the torque depends ultimately upon the field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ ([28]), this simplification is equivalent to assuming that the contribution of φ to $\boldsymbol{E}$ (cf. eq. (7) below) is large compared to that of $\boldsymbol{A}$ and that the contribution to the torque of terms involving $\boldsymbol{B}$ ($=$ curl $\boldsymbol{A}$) is negligible. The neglect of $\boldsymbol{A}$ will now be given heuristic justification. First, the gauge relation

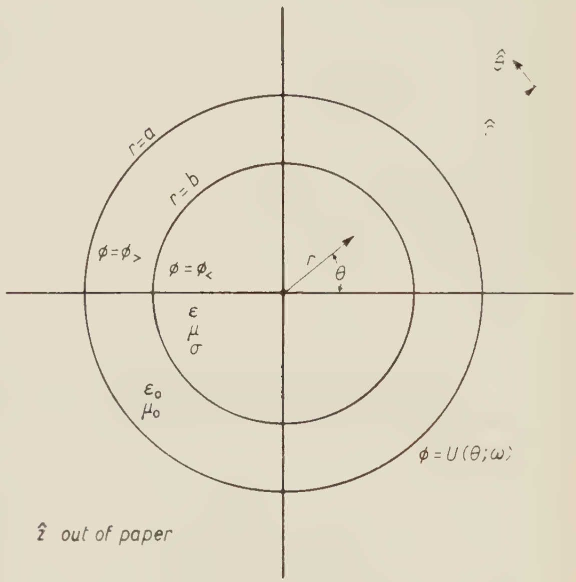

Fig. 1. – The geometry of the problem.

$$\operatorname{div} \boldsymbol{A} = -[\hat{\mu}\hat{\sigma} + j\omega\hat{\mu}\hat{\varepsilon}]\varphi \tag{6}$$

will be chosen; the use of the circumflex has been adopted merely to avoid specifying to which region the constitutive parameter refers. The choice of this gauge ([28]) renders the vector wave equation homogeneous so that any $\boldsymbol{A}$ present derives from surface currents at $r=a$ and $r=b$; if the driving frequency is sufficiently low, these should be negligible. Second, the vector potential will make itself felt in part through its contribution to the electric field:

$$\boldsymbol{E} = -\operatorname{grad}\varphi - j\omega\boldsymbol{A}\,. \tag{7}$$

Using the notion that, for a smoothly varying function, a spatial derivative can be very roughly approximated by

$$\frac{\mathrm{d}}{\mathrm{d}x_j} \sim \frac{1}{\varkappa}\,, \tag{8}$$

where x_j is some spatial co-ordinate and $\varkappa$ a characteristic distance, it is seen that away from $r=b$ the eq. (6) reduces to

$$|\boldsymbol{A}| \sim |\varphi|\,\varkappa\,|\hat{\mu}\hat{\sigma} + j\omega\hat{\mu}\hat{\varepsilon}| \tag{6'}$$

and the eq. (7) to (*)

$$|\boldsymbol{E}| \sim |\varphi|\,1/\varkappa + \omega\,|\boldsymbol{A}|\,. \tag{7'}$$

This means that $\mathscr{R}$, the ratio of the vector to the scalar contributions to $\boldsymbol{E}$, is given by

$$\mathscr{R} \sim \omega\varkappa^2\,|\hat{\mu}\hat{\sigma} + j\omega\hat{\mu}\hat{\varepsilon}|\,. \tag{9}$$

For the present problem one can set $\varkappa \sim b$, $\hat{\mu} \sim \mu'$, $\hat{\sigma} \sim \sigma'$ and $\hat{\varepsilon} \sim \varepsilon'$ to discover a rough condition for the negligibility of $\boldsymbol{A}$ in the determination of $\boldsymbol{E}$:

$$\omega^2 b^2 \mu'\varepsilon' \sqrt{1 + (\sigma'/\omega\varepsilon')^2} \ll 1\,. \tag{10}$$

The eq. (10) is effectively a near zone condition. Further, it can be shown that the contributions of $\boldsymbol{E}$ to the torque outweigh those of $\boldsymbol{B}$ when (10) is satisfied. Thus, if the eq. (10) is satisfied and if the angular distribution of potential over $r=b$ is chosen so as to minimize surface currents, the choice

(*) The replacing of a spatial operator such as grad or div, by $1/\varkappa$ in an equation of this sort will lead to a relation from which it is possible to estimate roughly the relative importance of the terms of the spatial derivative as compared to the other terms.

of the gauge (6) should enable one to deal solely with the scalar potential (electric field) in solving this problem.

Under the assumptions (6), (8) and (10) the scalar wave equation,

$$\nabla^2\varphi - j\omega\hat{\sigma}\hat{\mu}\varphi + \hat{\varepsilon}\hat{\mu}\omega^2\varphi = 0\,, \tag{11}$$

reduces to the Laplace equation

$$\nabla^2\varphi = 0\,. \tag{12}$$

That is, when the condition (10) is fulfilled, the space derivative terms of the scalar wave equation will be much more important than the others. The appropriate boundary conditions are [28]:

$$\varphi_> = U(\theta;\,\omega) \qquad r = b \tag{13a}$$

$$\varphi_< = \varphi_> \qquad r = a \tag{13b}$$

$$\xi\,\frac{\partial\varphi_<}{\partial r} = \varepsilon_0\,\frac{\partial\varphi_>}{\partial r}\,, \qquad r = a \tag{13c}$$

$$\varphi_< \text{ finite} \qquad r = 0\,. \tag{13d}$$

The symbol ξ in (13*c*) is defined by

$$\xi = \varepsilon_e - j\sigma_e/\omega\,, \tag{14}$$

where, σ'' being neglected,

$$\varepsilon_e = \varepsilon'\,, \tag{14a}$$

$$\sigma_e = \sigma' + \omega\varepsilon''\,. \tag{14b}$$

It can be seen from (14*b*) that the one parameter σ_e incorporates into the problem the conductivity effects considered by LAMPA [8] and the dipole rotation effects dealt with by BORN [14].

In general, the torque experienced by a body in an electromagnetic field is given by a summation of four integrals which involve the electric and magnetic field vectors and the essential densities of current and charge [28]. For this problen in which the magnetic field is to be neglected the torque reduces to a single integral over the surface $r = a$, since the essential density of volume charge will be zero. This integral involves quantities expressed as functions of the time t rather than the angular frequency ω and will, therefore, vary in time. Fortunately, the rules of harmonic analysis are such that the

time average axial torque per unit length $\langle L\rangle$ can be expressed directly as an integral in terms of functions of ω. For the geometry of Fig. 1, this integral is

$$(15)\qquad \langle L\rangle = \frac{1}{2}\,RE\int_0^{2\pi} a^2\bar{\eta}(\omega)E_\theta^*\,\mathrm{d}\theta\,,$$

where $\bar{\eta}(\omega)$ is the essential density of surface charge on the dielectric cylinder and E_θ^* is evaluated at $r=a$. Since the medium $a<r<b$ is free space and since the medium $r<a$ is not a perfect conductor, the continuity equation for surface charge implies that [28]

$$(16)\qquad \bar{\eta}(\omega) = (\boldsymbol{\xi} - \varepsilon_0)E_r\,.$$

Thus, the near zone conditions being satisfied,

$$(17)\qquad \langle L\rangle = \frac{1}{2}\,RE\int_0^{2\pi}(\xi-\varepsilon_0)\,\frac{\partial\varphi_<}{\partial r}\left[\frac{\partial\varphi_<}{\partial\theta}\right]^* a\,\mathrm{d}\theta\,,$$

where the derivatives are to be evaluated at $r=a$ and, the scalar potential will satisfy the Laplace equation.

The solution for φ is readily accomplished by the well known technique of expanding the potentials in series of the form

$$(18)\qquad \varphi = \sum_{n=0}^{\infty}(r^n + D_n r^{-n})(E_n\cos n\theta + F_n\sin n\theta)$$

and choosing the unknown constants to satisfy the boundary conditions (13). The procedure is straigtforward and only the result will be given:

$$(19)\qquad \varphi_< = \sum_{n=0}^{\infty}\delta_n[C_n\cos n\theta + S_n\sin n\theta]r^n\,,$$

where

$$(19a)\qquad \delta_n = \frac{2b^{-n}}{1+\xi_r}\left[1+\gamma^{2n}\,\frac{1-\xi_r}{1+\xi_r}\right]^{-1},$$

$$(19b)\qquad \xi_r = \xi/\varepsilon_0\,,$$

$$(19c)\qquad \gamma = a/b\,.$$

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The substitutions

$$C_n = C'_n - jC''_n \tag{20a}$$

$$S_n = S'_n - jS''_n \tag{20b}$$

together with the eq. (19) reduce (17), after considerable simple manipulation, to

$$\langle L \rangle = 4\pi\varepsilon_0 \left[\frac{\sigma_e/\varepsilon_0\omega}{|1+\xi_r|^2} \right] \sum_{n=0}^{\infty} n^2 \frac{\gamma^{2n}}{|1+\gamma^{2n}[(1-\xi_r)/(1+\xi_r)]|^2} [C'_n S''_n - C''_n S'_n] , \tag{21}$$

which is the desired formula for the average axial torque. The units of $\langle L \rangle$ in (21) are (farad-volt²/meter) or (kilogram-meter/s²) or (torque per unit length).

In the following section the eq. (21) will be reduced to a less complicated form by means of several reasonable simplifying assumptions, and the physical implications of this reduced form will be analyzed.

3. – Illustrative example.

The most common driving condition is the quadrant distribution; *i.e.* $U(\theta; t)$ is specified over four equally long and equally spaced arcs of the circle $r = b$. This configuration was first used by ARNO [2], and, in various modifications, has been used by almost all subsequent investigators. For this paper the driving potential is taken to be of the form

$$U(\theta; t) = \begin{cases} V \cos \omega t & 0 < \theta < \pi/2 & (22a) \\ V \sin \omega t & \pi/2 < \theta < \pi & (22b) \\ -V \cos \omega t & \pi < \theta < 3\pi/2 & (22c) \\ -V \sin \omega t & 3\pi/2 < \theta < 2\pi . & (22d) \end{cases}$$

A driving potential of this form will obviously produce a rotating electric field. The eq. (22) taken together with (1), (4), and (5) imply that

$$C_0(\omega) = S_0(\omega) = 0 , \tag{23a}$$

$$C_n(\omega) = \sqrt{2}\,\frac{V}{n\pi}\,(1-\cos n\pi)\sin\frac{n\pi}{2}\exp[j\pi/4] , \qquad n \geqslant 1, \tag{23b}$$

$$S_n(\omega) = \sqrt{2}\,\frac{V}{n\pi}\,(1-\cos n\pi)\exp[-j\pi/4] . \qquad n \geqslant 1. \tag{23c}$$

This implies that

$$C'_n S''_n - C''_n S'_n = \frac{2V^2}{n^2\pi^2}(1-\cos n\pi)^2 \sin\frac{n\pi}{2}, \tag{24}$$

and

$$\langle L\rangle = 32\frac{V^2}{\pi}\varepsilon_0\left[\frac{\sigma_e/\varepsilon_0\omega}{|1+\xi_r|^2}\right]\sum_{n=0}^{\infty}\frac{(-1)^n\gamma^{2(2n+1)}}{|1+\gamma^{2(2n+1)}(1-\xi_e)/(1+\xi_r)|^2}. \tag{25}$$

The convergence of this series is extremely rapid for $\gamma^2 \ll 1$. Hence, if $\gamma^2 \ll 1$, it is possible to approximate the torque by

$$\langle L\rangle = 32\frac{a^2\varepsilon_0}{\pi}\left(\frac{V}{b}\right)^2 \Gamma_T(f), \tag{26}$$

where

$$\Gamma_T(f) = \frac{\sigma_e/\varepsilon_0\omega}{|1+\xi_r|^2}. \tag{27}$$

It can be seen at once from eq. (26) that the torque varies linearly in the cross-sectional area and quadratically in the applied field strength; this is in accord with the findings of LAMPA [8] and BORN [14]. The variation of the torque with frequency is somewhat more complicated.

By means of (3a), (3b), (14a), and (14b) the eq. (27) can be reduced to

$$\Gamma_T(f) = \frac{\left(\dfrac{\sigma'\tau}{\varepsilon_0}\right)\dfrac{1}{\omega\tau} + (K_s - K_\infty)\dfrac{\omega\tau}{1+\omega^2\tau^2}}{\left[(1+K_s) - (K_s-K_\infty)\dfrac{\omega^2\tau^2}{1+\omega^2\tau^2}\right]^2 + \left[\left(\dfrac{\sigma'\tau}{\varepsilon_0}\right)\dfrac{1}{\omega\tau} + (K_s-K_\infty)\dfrac{\omega\tau}{1+\omega^2\tau^2}\right]^2}, \tag{27T}$$

$$K_s = \varepsilon_s/\varepsilon_0, \tag{28a}$$

$$K_\infty = \varepsilon_\infty/\varepsilon_0. \tag{28b}$$

The formula (27T) yields, when $\tau = 0$,

$$\Gamma_L(f) = \frac{\sigma'/\varepsilon_0\omega}{[1+K_s]^2 + [\sigma'/\varepsilon_0\omega]^2}. \tag{27L}$$

The frequency variation displayed by (27L) is very similar to that derived by LAMPA [8] for a spherical volume. $\Gamma_L(f)$ has its maximum value at a frequency given by

$$\omega_L = \frac{\sigma'}{\varepsilon_0(1+K_s)}. \tag{29}$$

The formula ($27T$) yields the Born type torque when $\sigma' = 0$:

$$(27B)\quad \Gamma_B(f) = \frac{(K_s - K_\infty)\,\dfrac{\omega\tau}{1+\omega^2\tau^2}}{\left[(1+K_s) - (K_s - K_\infty)\,\dfrac{\omega^2\tau^2}{1+\omega^2\tau^2}\right]^2 + \left[(K_s - K_\infty)\,\dfrac{\omega\tau}{1+\omega^2\tau^2}\right]^2},$$

It can be shown (29) that, in a rough approximation,

$$(30a)\qquad \tau \sim \chi_1 \frac{\eta}{kT} v_0\,,$$

$$(30b)\qquad \frac{\mu^2}{(kT)^2} \sim \chi_2 \frac{v_0}{kT}\,,$$

where v_0 is the volume of a fluid molecule, μ its intrinsic dipole moment, χ_1 and χ_2 are constants, and η the macroscopic viscosity of the liquid. Hence, if $\omega^2\tau^2 \ll 1$,

$$(31)\qquad \Gamma_B(f) \sim \chi_3 \left(\frac{\mu}{kT}\right)^2 \eta\omega\,.$$

The eq. (31) and (26) taken together yield a torque equation similar to that found by Born (14). The numerator of ($27B$) displays a frequency dependence similar to that derived by Debye (17). The denominator, absent in the expression of both Debye and Born, arises from the fact that the derivation given here involves fields determined by the solution of a boundary value problem in a bounded region rather than fields of known value in an infinite medium; the derivations of Born and Debye yielded the torque on a unit volume of material which was imbedded in an infinite space of the same material. The formula presented here for the Born effect is thought to be superior, especially for experimental work, to those of Born and Debye since it deals with a fluid sample of finite cross-section rather than with a unit volume within an unbounded medium and since it is based upon a dielectric loss formulation which is essentially phenomenological and macroscopic rather than upon one which is theoretical and grounded upon a simplified microscopic model of the fluid. Thus ($27B$) involves the experimentally observable and experimentally defined relaxation time τ while, in essence, the formulations of Born and Debye derive times of relaxation which depend upon the macroscopic viscosity η, which, while it certainly affects the time required by a dipole to line up in an applied electric field, is basically a macroscopic variable and cannot be used safely to evaluate microscopic motion phenomena in a fluid—a point which is well illustrated by the still unsolved problem of the limiting equivalent conductance of electrolyte ions. The frequency ω_B at which

$\Gamma_B(f)$ attains its maximum is a complicated function of the several parameters involved in (27B); however, it will fall near the maximum of ε''.

To illustrate the frequency behavior of the torque a curve of $\Gamma_T(f)$ has been computed for n-butyl alcohol ($CH_3 . CH_2 . CH_2 . CH_2 . OH$) and is shown

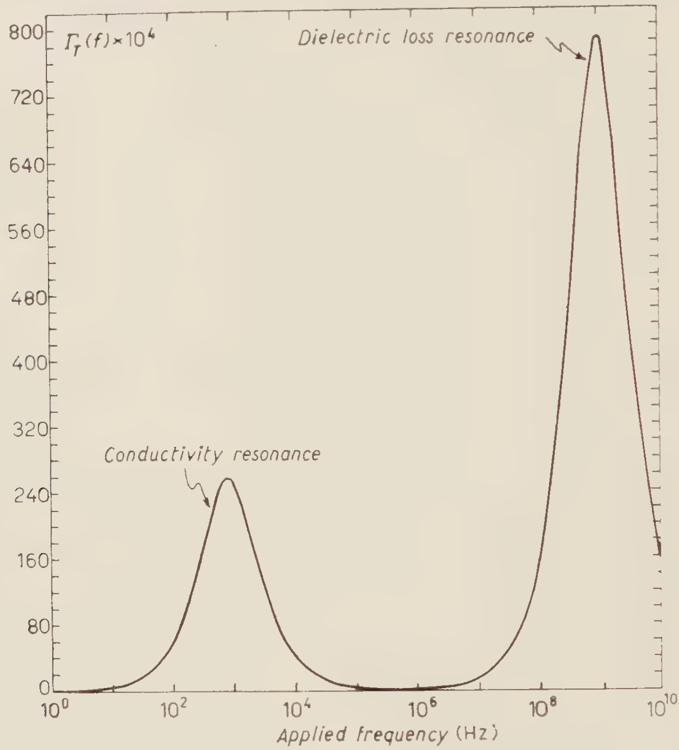

Fig. 2. – $\Gamma_T(f)$ *vs.* f for n-butyl alcohol.

in Fig. 2. The value of the conductivity ($9.12\cdot10^{-7}$ mho-meter at 30 °C) is that of HUNT and BRISCOE [30]. The values of K_s, K_∞, and τ were determined by DALBERT, MAGAT, and SURDUT [31] from data taken at 20 °C; they are

$$K_s = 18.5\,,$$

$$K_\infty = 3.8\,,$$

$$\tau = 0.663\cdot10^{-9}\ \mathrm{s}\,.$$

[30] H. HUNT and H. T. BRISCOE: *Journ. Phys. Chem.*, **33**, 1495 (1929).

[31] M.me DALBERT, M. MAGAT and A. SURDUT: *Bull. Soc. Chim. France*, **16***D*, 345 (1949).

The values of K_s, K_∞, and τ are probably correct to within 15%. The value of σ', due to the extreme difficulty of accurately measuring the conductivity of organic liquids, is of indeterminate accuracy. The data will serve, however, to illustrate, the general trends in the frequency variation of the torque. Moreover, the calculations have been extended as far as 10^{10} Hz in order to illustrate the behavior of $\Gamma_T(f)$ beyond its second resonance, although it would be extremely difficult to satisfy the near zone requirements (10) much beyond $3\cdot10^8$ Hz. The cross-over between the effects of conductivity and dielectric loss is shown in Fig. 3, in which $\Gamma_T(f)$, $\Gamma_L(f)$, and $\Gamma_B(f)$ are plotted over the frequency range $(1\cdot10^4 \div 1\cdot10^7)$ Hz.

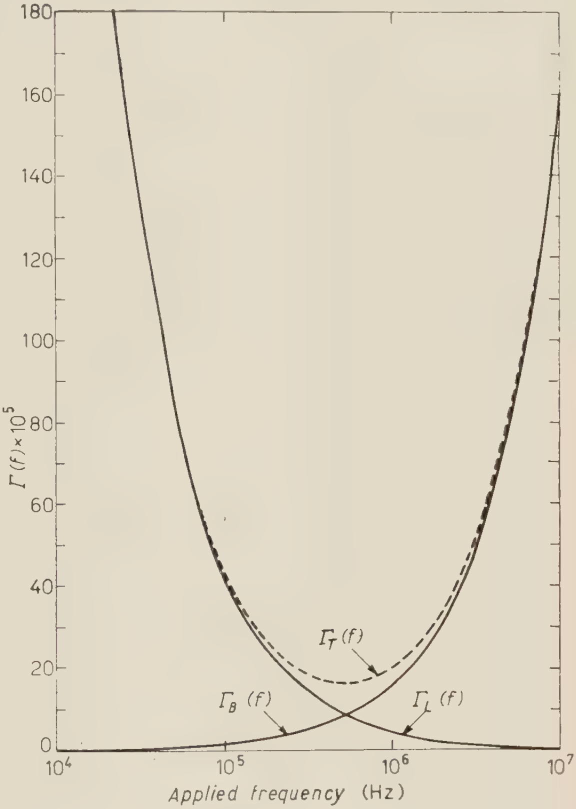

Fig. 3. – Variation of the gammas at cross-over for n-butyl alcohol.

For the example shown $\Gamma_T(f) \doteq \Gamma_L(f)+\Gamma_B(f)$. This is NOT a general rule and came about only because the two resonances were so far apart. If the conductivity of the alcohol had been somewhat larger, as it could easily have been for a sample which had not been carefully purified, the dip between the two resonances would have been much shallower and the effects of conductivity and dielectric loss not separable.

4. – Discussion and conclusions.

The results obtained in Sections **2** and **3** above bring out several points which it was desired to make. First, that in order to use the Laplace equation for the scalar potential and to neglect the vector potential it is necessary to satisfy a near zone condition—eq. (10) for the geometry of this paper. Second,

that both the Born and the Lampa-Lertes torque effects are basic consequences of energy loss phenomena in the dielectric fluid, and that, third, they are not in general separable. Finally, that the $\Gamma_T(f)$ curve will have two maxima, the first or conductivity resonance occurring near $f = \sigma'/2\pi(\varepsilon_0 + \varepsilon_s)$ and the second or dielectric loss resonance occurring near $f = 1/2\pi\tau$.

The geometry of Fig. 1 was chosen because cylindrical samples have been much used lately and because the Laplace equation is readily soluble in a concentric circle geometry. The experimental desirability of the set-up is questionable at higher frequencies since the assumed driving conditions cannot be maintained over a length of cylinder which is not short compared to a wavelength. If the cylinder is made short to avoid this problem, the effects of its ends and those of the center rod may become noticeable at higher frequencies. for these reasons it is felt that a spherical sample of dielectric is probably best, if it is correctly used. The problem of a sphere in a square box is difficult to handle mathematically unless the box is large compared to the sphere; but if it is, trouble may be encountered in maintaining the driving conditions at higher frequencies. Unfortunately, if the high-frequency driving problems are met by illuminating the sphere by plane waves from horns, one will be unable to make low frequency measurements without reconstructing the equipment. Probably the most versatile arrangement would be that of a spherical sample contained in a concentric spherical shell; the driving potential would be maintained over the surface of this shell. The concentric sphere problem is not over-complicated mathematically and should lead to a rapidly convergent series for the torque; by careful miniaturization it should be possible to satisfy the near zone conditions up to about $5 \cdot 10^8$ Hz in such a geometry.

The tendency of low molecular weight fluids to experience their dielectric loss resonance above 10^8 Hz means that it will not easily be possible to measure the far side of this resonance with a torque experiment of accuracy sufficient to permit a detailed comparison with simple theory. A failure to satisfy the near zone conditions implies the necessity of considering both scalar and vector potentials, strongly mediates in favor of microwaves horns to illuminate the sample, opens up the possibility of scattering resonances, and leads inexorably to torque formulae involving higher transcendental functions of complex argument. In view of the fact that the theory and experiment are still not in exact agreement at audio and radio frequencies, it is suggested that measurements above $3 \cdot 10^8$ Hz could profitably be omitted for the time being.

With regard to experiments at radio frequencies, the probability of obtaining agreement between experiment and theory to better than 10% seems very remote unless the experimental philosophy is changed. First, the number of polar fluids which have ever been purified sufficiently to be used in precise physico-chemical measurements, and especially those involving the electrical

conductivity, is not large. Second, the number of polar fluids for which ε_s, ε_∞, and τ are accurately known is not large. Third, the simple expressions for ε' and ε'', while adequate for the purposes of this paper and satisfactory for describing the behavior of numerous polar fluids of low molecular weight, are not exact and cannot, with high accuracy, be applied to fluids of intermediate and high molecular weight. Fourth, the approximations involved in neglecting $\boldsymbol{A}$ are basically not of quantitatively known worth and will begin to break down at higher frequencies. Finally, and most important of all, the electrical conductivity of an organic liquid is a highly unstable, unreproducible quantity (*) which depends critically on the purity of the liquid (**). In view of these factors it would appear that the most profitable mode of approaching the torque problem would be to measure the torque over the widest possible frequency range which is consistent with the near zone conditions, and then to seek that volume element in the (σ', ε_s, ε_∞, τ) four-space whose points best fit the observed data; this curve fitting process would, of course, be simplified somewhat if approximate values of the four parameters were known in advance. An approach of this sort would enable one to verify or reject the theory according as the volume element is of finite extent or null, and would possess the distinct advantage of requiring for its application no exact *a priori* knowledge of the four parameters. Moreover, the measurements of the four parameters so obtained would be electrodeless measurements. This would be of great importance in the measurement of σ' which is known to be influenced considerably by the nature of the electrodes.

In conclusion, the results of this paper chiefly indicate that:

i) The Born and Lampa-Lertes rotational effects are intimately connected with each other.

ii) A torque experiment can presumably be used to determine σ' and τ electrodelessly.

(*) An excellent example of this is acetone. The literature is replete with values of σ' for it; these values vary over several orders of magnitude. Two of the more interesting of the experiments to measure its conductivity can profitably be cited. Dippy and Hughes [32] showed that the electrical conductivity of acetone increases markedly upon standing and varies greatly with the purification techniques used. Eck [33] concluded that σ' depended upon so many factors, including the electrode material used in the measurement, that it was probably not possible to measure it.

(**) Since many organic liquids undergo degradation upon standing, being heated being exposed to light, etc., the purity and hence the conductivity can be expected to drift slowly but continuously.

[32] J. F. J. Dippy and S. R. C. Hughes: *Journ. Chem. Soc.*, 953 (1954).

[33] J. L. Eck: *Ann. Phys.*, **4**, 12 (1949).

* * *

The author wishes to express his thanks to Professor W. P. KING for his encouragement during the course of this research and for his valuable criticisms on the formulation of the near zone conditions.

Note added in proof.

It has been called to the author's attention that the problem of the torque on an object in a uniform, low frequency electric field was solved recently by means of the method of depolarizing factors [T. OGAWA: *Repts. Sci. Research Inst. Japan*, **36**, 408 (1960)] and the solution applied to the electrodeless measurement of the photoconductivity of CdS. The result of Ogawa, specialized to the case of a cylinder in a circularly polarized field, agrees with the result of this paper, specialized to the case of a field which is approximately uniform and circularly polarized [$\gamma \to 0$ and $U(\theta; t)$ given by (22)], except for a factor of $8/\pi^2$ which arises from the different driving conditions.

RIASSUNTO (*)

Presento un trattamento teorico unificato dell'effetto rotazionale Born-Lertes nel caso speciale di un sistema di due cilindri concentrici. Mostro che l'effetto Born e quello Lertes nascono naturalmente dai concetti di perdita dielettrica e di conducibilità effettiva. Do una curva teorica tipica della variazione della torsione in funzione della frequenza.

(*) *Traduzione a cura della Redazione.*

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The Invariant Amplitudes of Interaction Processes.

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Summary. — A systematic method of producing, where possible, invariants for a scattering process such that the associated amplitudes are free from kinematical singularities is derived. It is conjectured that such amplitudes cannot be found for most processes involving an odd number of photons. The results are proved completely in perturbation theory, but a method of proof independent of perturbation theory is outlined. Invariants for the elastic scattering of pions and deuterons, and nucleon Compton scattering are derived as examples. General interaction processes are also discussed.

1. – Introduction.

Recent work on the analyticity properties of scattering amplitudes, following the conjecture of Mandelstam [1] that the physical scattering amplitude is the boundary value of an analytic function of two complex variables has led to increasing interest in the associated problem of constructing amplitudes for processes with spin which have the same singularities as the spin 0 amplitudes. Evidently, the problem will become of greater importance as further work on production and higher order processes continues. It is expected that all results in this field will be proved or conjectured for the spin 0 case, and so the problem of producing amplitudes without kinematical singularities (that is, without more singularities than the analogous spin 0 case) for any order process remains.

It is the aim of this paper to provide a general framework within which kinematical-singularity-free amplitudes may be constructed, where possible,

(1) S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).

for a strong interaction process of any order. As interest is at present centred on scattering processes, however, the main part of this paper will consider this case, and the general case will be dealt with in a separate section. Isotopic spin is neglected throughout, as this only increases the multiplicity of the amplitudes and does not affect their singularities.

Our results are proved in perturbation theory. Although the quantitative results of this theory are untrue for strong interactions, we still believe that perturbation theory can correctly predict the analyticity properties of scattering functions. Furthermore, a method for showing the presence or absence of kinematical singularities in a given set of amplitudes which is independent of perturbation theory has been developed by WONG ([2]). As this method can only be applied to particular cases, however, it is beyond the scope of this paper to consider it, and the reader who doubts the analyticity predictions of perturbation theory is left to apply this method to any problem he chooses. Wong's method is summarized in the Appendix.

2. – Formulation of the problem.

We begin by defining, in the usual manner, the causal amplitude, M, for the process we are considering ([3]). For the moment, let us consider only interactions involving four particles with incoming momentum p_i. Momentum conservation is given by:

$$\sum_i p_i = 0 \quad (*) \tag{2.1}$$

and we define the three Lorentz scalars (**)

$$s_i = -(p_1 + p_{i+1})^2 \,. \tag{2.2}$$

In general, M will be a function of s_i, p_i and the spin parameters, β_i, of the theory. (We allow our definition of spin parameters to include polarization vectors, fermion spinors and γ matrices.)

([2]) M. L. GOLDBERGER, M. T. GRISARU, S. W. MACDOWELL and D. Y. WONG: *Phys. Rev.*, **120**, 2250 (1960).

([3]) See, for example, J. D. JACKSON in *Dispersion Relations* (Edinburgh, 1961).

(*) Summation convention: we assume the summation convention for Greek suffixes, but not for Latin.

(**) We set $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, although the results are, in general, independent of metric.

Lorentz invariance allows us to write M in the form:

$$M(s_i, p_i, \beta_i) = \sum_j A_j(s_i)\, M_j(p_i, \beta_i)\,, \tag{2.3}$$

where the M_j are an independent set of scalar functions which can be formed from the momenta and spin parameters, and which are invariant under parity, time reversal and charge conjugation. The number of independent functions required for a given problem can be decided by an analysis of helicity states, as suggested by JACOB and WICK [4], or by an analysis of possible angular momentum states. Throughout this paper, we call the A_j the amplitudes, and M_j the invariants.

We require that the A_j are free from kinematical singularities . Even this stringent restriction does not uniquely determine a set of M_j; in fact, it does not even ensure that such a set of M_j exist. However, we know that there are no kinematical singularities present in the amplitude for a spin 0 process, so let us take a given perturbation graph for any process and write down its contribution to the causal amplitude. The Feynman integral for this graph differs from the integral for an analogous spin 0 graph only in the appearance in the numerator (of the integrand) of spinors, γ matrices, polarization vectors and momenta—the denominators have exactly the same form. The integration of any such integral to give us the desired contribution does not introduce any further singularities other than those already present in the denominator. Thus, if we write M in the form:

$$M = \sum_j B_j(s_i)\, N_j(p_i, \beta_i)\,,$$

where the sum is taken over *all* possible invariants which one can write down from every graph in every order of perturbation theory, then we are assured, in perturbation theory at any rate, that a set of B_j free from kinematical singularities exists. I am indebted to Mr. J. NUTTALL for pointing this out.

It is therefore sufficient to show that the reduction of every possible invariant to our basic independent set M_j produces no singularities in order to show that our A_j are free from kinematical singularities. In other words, if we can write:

$N_j = \sum_l K_{jl} M_l$, where K_{jl} is free from pole and branch point singularities [*]

then, since

$$M = \sum_j B_j N_j = \sum_{j,l} B_j K_{jl} M_l\,,$$

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[4] M. JACOB and G. C. WICK: *Ann. Phys.*, 7, 404 (1959).

[*] K_{jl} is not a square matrix, since there are, of course, many more N's than M's.

with B_j a set free from kinematical singularities, we have $A_l = \sum_j B_j K_{jl}$, and the A_j are free from kinematical singularities.

Alternatively, we say that a kinematical singularity arises when the reduction of a possible invariant to the basic set produces a singularity. For example, if p_1, p_2, q_1, q_2, are the initial and final nucleon and pion incoming momenta for pion-nucleon scattering, it is well known ([5]) that the invariants 1 and $i\gamma \cdot Q$, where $Q = \frac{1}{2}(q_1 - q_2)$, give kinematical-singularity-free amplitudes. Let us now take the set $\gamma \cdot p_1 \gamma \cdot Q$ and $i\gamma \cdot Q$ instead. Since $1\,(=(\gamma \cdot p_1 \gamma \cdot Q/2p_1 \cdot Q) - (im\gamma \cdot Q/2p_1 \cdot Q))$ is a possible invariant, kinematical singularities are present in the second set. Obviously, the set $\gamma \cdot p_1 \gamma \cdot Q/p_1 \cdot Q$ and $i\gamma \cdot Q/p_1 \cdot Q$ have amplitudes with no kinematical singularities, but as they have singularities themselves we must discard them. A kinematical singularity in the amplitude, then, does not imply that the invariant has a zero at the singularity. If this were the case, we could simply divide the invariant by the singularity producing term, and a bounded invariant, suitable for the problem, may result.

One major problem arises immediately. In order to prove that a set of amplitudes is free from kinematical singularities, we propose to show that the reduction of all invariants to a given set produces no singularities. In a process involving photons, however, the sum of all graphs of a given order is gauge-invariant, whereas the general graph is not. Since our basic set will be gauge-invariant, we can only reduce gauge-invariant terms, not the general term, to our basic set. We shall therefore consider photon processes separately.

3. – Invariants for four-particle processes.

3·1. – We now develop a general method for forming an independent set of invariants without kinematical singularities in the associated amplitudes for a scattering process not involving photons. As explained in the previous section, we may write the causal amplitude in the form:

$$M = \sum_j A_j(s_i)\, M_j(p_i, \beta_i)\,. \tag{2.3}$$

Lorentz invariance also allows us to write this in the more tractable form:

$$M = \sum_{j,k} A^n_{jk}(s_i)\, C^{(n)}_j(p_i)\, L^{(n)}_k(\beta_i)\,. \tag{3.1}$$

This follows immediately from the fact that the causal amplitude is a multilinear form in spinors and polarization vectors. By $L^{(n)}_k$ (for fixed k) we mean

([5]) See, for example, G. F. CHEW: *Double Dispersion Relations and Unitarity as the Basis for a Dynamical theory of Strong Interactions*, UCRL preprint 9289 (1960).

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a covariant tensor $L_{k,\mu\nu\ldots}$ of order n, and with each $L_k^{(n)}$ we associate a set of momentum functions $C_j^{(n)}$ which are contravariant tensors $C_j^{\mu\nu\ldots}$ of the same order. For a given scattering problem, the L-functions will be unique in our formalism. We write the sixteen irreducible γ matrices as five tensor forms; 1, $i\gamma_\mu$, γ_5, $i\gamma_5\gamma_\mu$, $\gamma_\mu\gamma_\nu$, with $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$ and the i included as a consequence of charge-conjugation invariance. Therefore, for processes not involving fermions, the set L_k has one component, for processes with two fermions, five components, and for processes with four fermions, fifty components (*) (twenty-five from crossing the two spin spaces, and double this since two spinors can be interchanged). The inclusion of a spin 1 particle, whose spin is described by a polarization vector, ε_i, does not alter the number of components of L, but merely increases their tensor order.

For example, the components of L for spin 1, fermion elastic scattering are:

$$u\varepsilon_{2\nu}\varepsilon_{1\mu}u\,;\quad u\varepsilon_{2\nu}i\gamma_\varrho\varepsilon_{1\mu}u\,;\quad u\varepsilon_{2\nu}\gamma_5\varepsilon_{1\mu}u\,;\quad u\varepsilon_{2\nu}\gamma_5 i\gamma_\varrho\varepsilon_{1\mu}u\,;\quad u\varepsilon_{2\nu}\gamma_\varrho\gamma_\sigma\varepsilon_{1\mu}u\,;$$

the notation is obvious.

On the other hand, the choice of possible C_j is far from unique. Ideally, we would like to span C^n with normalized, orthogonal vectors, and, in fact such a set may be constructed for any problem. For example,

$$P_1=p_3+p_4+\frac{E_3+E_4}{E_2}p_2\,;\qquad P_2=p_3-p_4+\frac{E_3-E_4}{E_2}p_2\,;\qquad P_3=\varepsilon^{\mu\varkappa\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}\,;$$

where E_i is the barycentric energy of p_i for the channel $1+2\to3+4$, have zero time component and $P_4=p_1+p_2$ has zero space component in the barycentric system mentioned. Hence, $\hat{P}_i=P_i/|P_i^2|^{\frac{1}{2}}$ form a set of normalized orthogonal vectors bounded in the given barycentric system, and so the Lorentz invariance of our theory ensures that any scalar formed from them is bounded in every frame. Hence, if we write a general term of $C^{(n)}$ in the form $\hat{P}_i^\mu\hat{P}_j^\nu\ldots\hat{P}_k^\varrho$, it is easy to see that the reduction of any invariant to this basic set will lead to no pole singularities. However, branch point singularities are introduced by our normalization, and so, although it is possible to build up a complete theory of invariants for any order interaction using this method, it will not lead to the required kinematical-singularity-free formalism.

(*) The requirements of representation theory limit this number to less than 50, but as the fermion-fermion problem has been solved completely (2), we shall not labour this point.

Alternatively, we can develop the bases of the C's in terms of the given momenta and the isotropic tensors $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\varrho\sigma}$, using as few momenta as possible. Since all higher order isotropic tensors are a product of $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\varrho\sigma}$ it is easy to ensure that no simpler tensor has been omitted, one tensor being « simpler » than another if it contains fewer momenta. One easy method of forming them is to write down all products of the form:

$$Q_i'^{\mu} Q_j'^{\nu} \dots Q_k'^{\varrho} ,$$

where

$$Q_i' = p_i \qquad (i = 1, 2, 3)$$

$$Q_4' = \varepsilon^{\mu\alpha\beta\gamma} p_{1,\alpha} p_{2,\beta} p_{3,\gamma} .$$

Reducing as many terms as possible to simpler forms, appropriate bases can be formed. We note possible bases for such C functions. Only the first few terms of the higher order bases are given, and the other terms, if needed, can easily be found.

$C^{(0)}$: $\quad 1$

$C^{(1)}$: $\quad p_i^{\mu}; \quad \varepsilon^{\mu\nu\varrho\sigma} p_{1,\nu} p_{2,\varrho} p_{3,\sigma} ,$

$C^{(2)}$: $\quad g^{\mu\nu}; \; p_i^{\mu} p_j^{\nu}; \; \varepsilon^{\mu\nu\varrho\sigma} p_{i,\varrho} p_{j,\sigma} \;\; (i<j); \; (p_i^{\mu} \varepsilon^{\nu\alpha\beta\gamma} p_{1,\alpha} p_{2,\beta} p_{3,\gamma} - p_i^{\nu} \varepsilon^{\mu\alpha\beta\gamma} p_{1,\alpha} p_{2,\beta} p_{3,\gamma}),$

$C^{(3)}$: $\quad g^{\mu\nu} p_j^{\varrho}; \quad g^{\mu\varrho} p_j^{\nu}; \quad g^{\nu\varrho} p_j^{\mu}; \quad \varepsilon^{\mu\nu\varrho\sigma} p_{j,\sigma}; \quad g^{\mu\nu} \varepsilon^{\varrho\alpha\beta\gamma} p_{1,\alpha} p_{2,\beta} p_{3,\gamma};$

$$p_k^{\varrho} \varepsilon^{\mu\nu\alpha\beta} p_{i,\alpha} p_{j,\beta} \;\; (i<j); \quad p_i^{\mu} p_j^{\nu} p_k^{\varrho}, \text{ etc.}$$

$C^{(4)}$: $\quad \varepsilon^{\mu\nu\varrho\sigma}; \; g^{\mu\nu} g^{\varrho\sigma}; \; g^{\mu\varrho} g^{\nu\sigma}; \; g^{\mu\sigma} g^{\nu\varrho}; \; g^{\mu\nu} p_i^{\varrho} p_j^{\sigma}; \; g^{\mu\varrho} p_i^{\nu} p_j^{\sigma} \;\; (i \times j \neq 1) \; ; \; p_i^{\mu} p_j^{\nu} p_k^{\varrho} p_l^{\sigma} ,$

$$(i, j, k, l = 1, 2, 3).$$

3.2. *Physical constraints.* – Having defined suitable bases for the C and L functions, we can now write down the invariants which result from taking the direct product of the C and L bases, and apply the physical constraints, which fall into two classes (*).

First, we have the spin constraints. For fermions, the Dirac equation:

$$(i\gamma \cdot p_i + m_i) u_i = 0 \tag{3.2}$$

(with p_i the momentum of a fermion of mass m_i and with spinor u_i), and the two identities:

$$\gamma_\mu \gamma_\nu \gamma_\varrho = g_{\mu\nu} \gamma_\varrho - g_{\mu\varrho} \gamma_\nu + g_{\nu\varrho} \gamma_\mu + \gamma_5 \varepsilon_{\mu\nu\varrho\sigma} \gamma^{\sigma} , \tag{3.3}$$

$$\gamma_\mu \gamma_\nu = 2 g_{\mu\nu} - \gamma_\nu \gamma_\mu \tag{3.4}$$

(*) It should be remembered when performing these reductions that the conservation of momentum eq. (2.1) can be applied to reveal dependent terms.

will enable us to reduce many terms to those already present. For spin 1 particles with momentum p_i and polarization ε_i, reductions may be made by using the polarization condition $p_i \cdot \varepsilon_i = 0$.

The application of the TCP constraints, the second class of constraints, to reduce the number of invariants, has been described by many authors ([6]), and so will not be discussed in detail here. Essentially, we find that a linear combination of the terms remaining is necessary in order to satisfy them. Often, charge-conjugation or time-reversal require that only certain combinations, $\mathscr{P}_i$, of the momenta may occur. In this case, it may be useful to write the C functions in terms of the $\mathscr{P}_i$ rather than the p_i ([*]).

Our reductions are now complete, and we conjecture that the right number of invariants will always result by this procedure, since we have applied all known constraints to a formalism which produces invariants which are essentially independent outside these constraints. It now remains to prove that our formalism produces invariants with kinematical-singularity-free amplitudes. We first observe that the application of the physical constraints has obviously not introduced any singularities. Similarly, it is easy to see that our choice of L functions has introduced no singularities, *i.e.*, that the reduction of any invariant from another L basis does not introduce singularities. The only different L basis we could choose would contain more γ matrices. However, these may always be reduced to our basic set by the identity (3.3) and the straightforward reduction of $\gamma_5\gamma_\mu\gamma_\nu$. These reductions obviously introduce no singularities, and so the only singularities which could be present must be in the C functions.

To prove that singularities are absent in the reductions to the given C bases, we need only consider the reduction of those tensors containing $\varepsilon^{\mu\nu\varrho\sigma}$ either once or not at all, since the identity ([**]):

$$\varepsilon^{\mu\nu\varrho\sigma}\varepsilon_{\alpha\beta\gamma\delta} = -\begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\alpha & \delta^\varrho_\alpha & \delta^\sigma_\alpha \\ \delta^\mu_\beta & \delta^\nu_\beta & \delta^\varrho_\beta & \delta^\sigma_\beta \\ \delta^\mu_\gamma & \delta^\nu_\gamma & \delta^\varrho_\gamma & \delta^\sigma_\gamma \\ \delta^\mu_\delta & \delta^\nu_\delta & \delta^\varrho_\delta & \delta^\sigma_\delta \end{vmatrix}, \tag{3.5}$$

enables us to reduce any other tensor to this form without singularities. An

([6]) See, for example, reference ([2]) above.

([*]) The requirements of the Pauli principle or crossing symmetry may also guide our choice of $\mathscr{P}_i$.

([**]) To avoid confusion, we define $\varepsilon_{\mu\nu\varrho\sigma}$ in the normal way, and define

$$\varepsilon^{\alpha\beta\gamma\delta} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\varrho}g^{\delta\sigma}\varepsilon_{\mu\nu\varrho\sigma}.$$

This is *minus* the usual definition of $\varepsilon^{\alpha\beta\gamma\delta}$.

inspection of the derived bases will indicate which tensors with one or no $\varepsilon^{\mu\nu\varrho\sigma}$ have been omitted, and it is a straightforward, though tedious, matter to show that reducing this finite set of tensors leads to no singularities. Identities such as

$$(3.6)\qquad p_1^\mu \varepsilon^{\nu\lambda\varrho\sigma} p_{1,\lambda} p_{2,\varrho} p_{3,\sigma} - p_1^\nu \varepsilon^{\mu\lambda\varrho\sigma} p_{1,\lambda} p_{2,\varrho} p_{3,\sigma} = - p_1^2 \varepsilon^{\mu\nu\varrho\sigma} p_{2,\varrho} p_{3,\sigma} + \\ + (p_3 \cdot p_1) \varepsilon^{\mu\nu\varrho\sigma} p_{2,\varrho} p_{1,\sigma} + (p_2 \cdot p_1) \varepsilon^{\mu\nu\varrho\sigma} p_{1,\varrho} p_{3,\sigma},$$

are useful in these reductions.

Thus the formalism we have produced gives us invariants with associated amplitudes free from kinematical singularities. There may be other ways of writing the C functions in order to achieve this, but the way shown seems to be the most useful for practical calculations.

As an example, let us produce invariants without singularities for the process $\pi + \mathrm{D} \to \pi + \mathrm{D}$. Denoting the initial and final incoming pion and deuteron momenta by q_1, q_2, d_1, d_2 respectively, and the initial and final deuteron polarization by ε_1, ε_2, the L function for this process is simply $\varepsilon_{2\nu}\varepsilon_{1\mu}$. If we now define the variables $D = \frac{1}{2}(d_1 - d_2)$, $Q = \frac{1}{2}(q_1 - q_2)$, and apply parity invariance and the polarization condition immediately, the only possible invariants are $\varepsilon_2 \cdot \varepsilon_1$, $D \cdot \varepsilon_2 D \cdot \varepsilon_1$, $Q \cdot \varepsilon_2 Q \cdot \varepsilon_1$, $D \cdot \varepsilon_2 Q \cdot \varepsilon_1$, $D \cdot \varepsilon_1 Q \cdot \varepsilon_2$. Time reversal tells us now that only the sum of the last two terms may occur, leaving us with the required four kinematical-singularity-free amplitudes.

3·3. *Processes involving photons.* – Let us now consider the case of interactions involving photons. The formalism explained above is obviously valid here, except that we have one further constraint to apply, namely, the gauge-invariance condition. We may write this in the form:

$$(3.7)\qquad M_\mu k_i^\mu = 0\,,$$

where $M_\mu \varepsilon_i^\mu$ is the causal amplitude, and k_i is the momentum of a photon with polarization ε_i. This constraint implies that the gauge-invariant invariants formed from the invariants, $\mathscr{M}_i$, for a general spin 1 process, in order to be bounded, must be written in the form:

$$(3.8)\qquad M_i = \sum_j \alpha_{ij} \mathscr{M}_j\,,$$

where the α_{ij} are functions of the s_i. If we now write the invariants for the spin 1 process in terms of these M_i and other non-gauge-invariant forms, we see that we can reduce any possible invariant for a photon process to this set, and that the gauge-invariance condition implies that the amplitudes asso-

ciated with the non-gauge-invariant forms must vanish. However, because of the form of our gauge-invariant invariants (3.8), we cannot perform our reductions without introducing kinematical singularities. It is therefore necessary to see if we can form our invariants in another way.

In only one case, the above formalism is sufficient to produce the desired amplitude, namely, when the overall matrix element is pseudoscalar, and the only particle with spin is one photon (*e.g.* $\gamma+\pi\to\pi+\pi$). Here only one amplitude is necessary, namely, $\varepsilon^{\mu\alpha\beta\gamma}p_{1,\alpha}p_{2,\beta}p_{3,\gamma}\varepsilon_{\mu}$, which is automatically gauge-invariant. In all other cases, some form of momentum normalization (or, at least, of division of the invariants (3.8) by a function of the s_i) with the proviso that bounded functions remain, is necessary.

When an even number of photons are involved, such a procedure is indeed possible. Consider first the case of two photons. Let k_1, k_2 denote the photon momenta and ε_1, ε_2 their polarization, and let the other two particles (which we shall assume have the same mass) have momenta p_1 and p_2. If we now define the vectors:

$$P=\tfrac{1}{2}(p_1-p_2)$$

$$K=\tfrac{1}{2}(k_1-k_2)$$

$$Q=k_1-k_2$$

$$P'=P-\frac{P\cdot K}{K^2}K\,,$$

$$N=\varepsilon^{\mu\alpha\beta\gamma}P'_\alpha K_\beta Q_\gamma\,,$$

then the vectors P', K, Q and N form a set of orthogonal vectors, and although $\hat{P}'$ (and $\hat{K}$) is unbounded at certain points, the scalars $\hat{P}'\cdot\varepsilon_i$ (and $\hat{K}\cdot\varepsilon_i$) are everywhere bounded in the barycentric system of the process $\gamma+X\to\gamma+X$ (*). Therefore, if we span with the vectors $\hat{P}'$ and $\hat{N}$ that part of the C bases which is contracted with the polarization vectors, and we span the remainder with p_1, p_2, p_3, $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\varrho\sigma}$ as before, we are assured that our invariants are bounded. It is also obvious that no pole kinematical singularities are present when fermions are absent, and the result may also be proved when fermions are present by a careful « reduction » of simpler invariants (*e.g.* $\varepsilon_1\cdot\varepsilon_2$, $\gamma\cdot\varepsilon_1\gamma\cdot\varepsilon_2$) to the given set. Branch point singularities are also absent, since, only the

(*) We can prove this as follows: Since Q and N are obviously bounded in the given barycentric system, and $g^{\mu\nu}=\hat{P}'^{\mu}\hat{P}'^{\nu}+\hat{N}^{\mu}\hat{N}^{\nu}+\hat{Q}^{\mu}\hat{Q}^{\nu}-\hat{K}^{\mu}\hat{K}^{\nu}$, we need only prove that $\hat{K}.\varepsilon_i$ is bounded. $K^2=s_3/4$, therefore $s_3\to 0$ as $K^2\to 0$. Hence $k_2=-k_1$ in the limit, and $K\cdot\varepsilon_1=8(k_1\cdot\varepsilon_1)/t$, which is bounded. Similarly for $\hat{K}\cdot\varepsilon_2$. This proof is due to GOURDIN and MARTIN (7).

(7) M. GOURDIN and A. MARTIN: *Nuovo Cimento*, **17**, 224 (1960).

normalized variables $\hat{P}'$ and $\hat{N}$ remain contracted with polarization vectors when the physical constraints are applied. Therefore the only normalization factors present are P'^2, N^2 and $(P'^2)^{\frac{1}{2}}(N^2)^{\frac{1}{2}} = P'^2K^2$, and all these are rational functions of the s_i. When four photons are present, we can produce a different set of vectors for each set of photons.

This procedure is obviously not applicable when an odd number of photons are involved. The symmetries introduced by elastic scattering are absent for such processes, and so the above variables do not produce bounded scalars. Furthermore, even if suitable normalized variables could be defined, we would still have branch point singularities in the amplitudes. It appears most probable, therefore, and, in fact, we conjecture that it is not possible to form kinematical-singularity-free amplitudes for such processes. However, BALL ([8]) has shown that, although one of the standard set of amplitudes for nucleon photoproduction possesses a kinematical singularity, it is possible to produce a subtracted dispersion relation for this amplitude with the desired properties, and this procedure may prove feasible in most cases.

As an example of a photon process for which we can produce kinematical-singularity-free amplitudes, let us consider nucleon Compton scattering, for which six amplitudes are required. The L functions for this process were given in Section **3·1**. Using the variables defined above, considerations of parity and time reversal limit us to the amplitudes: $\hat{P}'\cdot\varepsilon_1\hat{P}'\cdot\varepsilon_2$, $\hat{N}\cdot\varepsilon_1\hat{N}\cdot\varepsilon_2$, $(\hat{P}'\cdot\varepsilon_1\hat{N}\cdot\varepsilon_2 - \hat{P}'\cdot\varepsilon_2\hat{N}\cdot\varepsilon_1)\gamma_5$, $\hat{P}'\cdot\varepsilon_1\hat{P}'\cdot\varepsilon_2 i\gamma\cdot K$, $\hat{N}\cdot\varepsilon_1\hat{N}\cdot\varepsilon_2 i\gamma\cdot K$, $(\hat{P}'\cdot\varepsilon_1\hat{N}\cdot\varepsilon_2 + \hat{P}'\cdot\varepsilon_2\hat{N}\cdot\varepsilon_1)\gamma_5 i\gamma\cdot K$. (The spinors having been suppressed.) In fact, these are similar to the set of invariants produced by PRANGE ([9]) for electron Compton scattering, the only difference being that PRANGE did not normalize his momentum variables, and so his amplitudes had kinematical singularities.

4. – Invariants for general interactions.

The main features of this formalism can be applied to a process of any order. The L functions are again unique, and the constraints remain valid. However, one must redefine the bases of the C functions in terms of the momenta available in the process. In the case of five-particle interactions, we now have four independent momenta, and so the space of the C functions can be spanned by either tensors or pseudotensors. Obviously, we can write the tensors in one base in terms of the pseudotensors in the other, so, if we require C functions for a fermion process, we must span a third C basis by picking

([8]) J. S. BALL: *The Application of the Mandelstam Representation to Photoproduction of Pions from Nucleons*, UCRL preprint 9172 (1960).

([9]) R. E. PRANGE: *Phys. Rev.*, **110**, 240 (1958).

such tensors and pseudotensors which are independent and have the minimum number of momentum variables, since we can decide the parity of the term by the appropriate choice of γ matrix. When photons are present, the considerations of the last section will apply. We note possible tensor and pseudotensor C bases for $n = 0, 1, 2$.

$n =$	tensor base	pseudotensor base	
0	1	$\varepsilon^{\alpha\beta\gamma\delta}p_{1,\alpha}p_{2,\beta}p_{3,\gamma}p_{4,\delta}$	
1	p_i^{μ}	$\varepsilon^{\mu\alpha\beta\delta}p_{k,\alpha}p_{j,\beta}p_{i,\gamma}$	$k<j<i=1 \dots 4$
2	$g^{\mu\nu}$; $p_i^{\mu}p_j^{\nu}$	$\varepsilon^{\mu\nu\alpha\beta}p_{j,\alpha}p_{i,\beta}$	$j<i=1 \dots 4$
	$i = 1 \dots 4$	$p_l^{\mu}\varepsilon^{\nu\alpha\beta\gamma}p_{k,\alpha}p_{j,\beta}p_{i,\gamma}$	$k<j<i=1 \dots 4$
	$j = 1 \dots 3$		$\varepsilon_{ijkl} \neq 0$

When six or more particle interactions are considered, one is immediately appalled by the vast amount of algebra necessary to describe even the spin 0 case. We now have n momentum variables which may be reduced to $n-1$ by conservation of momentum. However, these cannot all be independent, because only four independent vectors can exist in Lorentz four-space. The linear relationship between the momenta is a complicated function of the s variables ([10]), and so it is impossible to find a set of C functions if $n > 5$; once we decide on four momenta with which to span $C^{(1)}$, even writing the other momenta in terms of these will introduce singularities. The only consistent method in this case, (and perhaps also in the five particle case, for more complicated problems) seems to be to use the normalized variables suggested in Section 3·1. Although these will introduce branch-point singularities, at least we know where the singularities are, and may be able to handle them.

5. – Summary.

Let us now summarize our general procedure for producing wherever possible a set of kinematical-singularity-free amplitudes for any order interaction process:

a) Write down the appropriate L (spin) basis.

([10]) For example, G. Källén and B. H. Wilhelmsson: *Generalized Singular Functions*, Nordita preprint (1958), discuss the kinematics of six-particle scattering.

([11]) D. Hall and A. S. Wightman: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **31**, No. 5 (1957).

b) Choose a suitable set of C (momentum) functions, taking into account the requirements of the physical constraints.

c) Write down the terms obtained by crossing the C and L bases, and apply the following constraints:

i) If spin $\frac{1}{2}$ particles are present, remove the terms reduced by the application of the Dirac equation and the identities (3.3) and (3.4).

ii) If spin 1 particles are present, apply the condition $p \cdot \varepsilon = 0$.

iii) If photons are also present, use the procedure explained in the text

iv) Apply parity, charge-conjugation and time-reversal invariance.

v) Finally, check to see that the conservation of momentum equation does not imply that some terms are not independent.

We now have a set of invariants which have associated amplitudes with no kinematical singularities.

* * *

I wish to thank Professor S. TREIMAN for much help and advice during the whole of this work, also Drs. E. LEADER and J. C. TAYLOR for many useful discussions. My thanks are also due to Professor J. HAMILTON and Dr. J. C. POLKINGHORNE for continued support and advice, and to the Shell International Petroleum Company Limited for a research scholarship.

APPENDIX

We present in this appendix a brief account of the method of WONG ([2]) for showing the presence or absence of kinematical singularities in any set of invariants. We begin by writing the causal amplitude, M, in the form:

$$M = M_{\mu\nu\ldots\varrho}\varepsilon_i^\mu \varepsilon_j^\nu \ldots \varepsilon_k^\varrho ,$$

where $M_{\mu\nu\ldots\varrho}$ now contains no polarization vectors. We now define

$$\mathcal{N}_i = \text{Trace}\,(O_i^{\mu\nu\ldots\varrho} M'_{\mu\nu\ldots\varrho}) ,$$

where M' is M with each spinor replaced by the corresponding positive or negative energy projection operator and O_i is any kinematical-singularity-free

function of the same tensor order as $M_{\mu\nu\dots\varrho}$ satisfying $O_k^{\mu}\varepsilon_{i,\mu} = O_k^{\nu}\varepsilon_{j,\nu} = \dots = 0$. The Hall-Wightman theorem (11) now tells us that $\mathcal{N}_i$ has no more singularities than M.

Writing $M = \sum_{j=1}^{n} A_j M_j$, and forming n independent $\mathcal{N}_i$, we may write:

$$\mathcal{N}_i = \sum_{j=1}^{n} D_{ij} A_j \,.$$

The zero singularities of D_{ij} indicate possible singularities of A_j. Having found these we must look at a suitable partial wave expansion of the A_j, or otherwise, and decide whether, in fact, singularities are present at these points in the A_j.

RIASSUNTO (*)

Derivo un metodo sistematico per produrre, quando è possibile, invarianti per un processo di scattering tali che le ampiezze associate siano prive di singolarità cinematiche. Congetturo che tali ampiezze non si possano trovare per molti processi che coinvolgono un numero dispari di fotoni. I risultati vengono esaurientemente dimostrati nella teoria della perturbazione, ma si delinea un metodo di prova indipendente dalla teoria della perturbazione. Come esempio deduco gli invarianti per lo scattering elastico di pioni e deutoni e per lo scattering Compton dei nucleoni. Discuto anche i processi di interazione generici.

(*) *Traduzione a cura della Redazione.*

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Two-Point Function and Generalized Free Fields.

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Summary. — Several theorems are proven which relate to the possibility of constructing a noninteracting field with an arbitrary two-point Wightman function. They are: (a) if $\varphi(x)$ is a complete local field, and $[\varphi(x), \varphi(y)] = D(x-y)$, where D is an arbitrary operator depending on x and y only through their difference, then D is a c-number function; (b) such fields are generalized free fields, as defined by Greenberg; (c) any generalized free field is unitarily equivalent to a superposition of Klein Gordon fields, and moreover the asymptotic condition and unitarity restrict this to a superposition of ordinary fields with different discrete masses.

Introduction.

In this paper a local complete field $\varphi(x)$ is investigated that satisfies the restriction,

$$[\varphi(x), \varphi(y)] = D(x-y)\,, \tag{1}$$

where D is some arbitrary operator. A proof is given in Section **1** that D si necessarily a c-number. In Section **2** it is shown that such fields are generalized free fields as defined by GREENBERG (1). The question of whether or not a generalized free field is simply a superposition of ordinary free fields is discussed in Section **3**.

(*) National Science Foundation Predoctoral Fellow.

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(1) O. W. GREENBERG: *Bull. Amer. Phy. Soc.*, **6**, 306, (1961).

Assumptions: We will deal throughout with a single complete, local, scalar and hermitian field, $\varphi(x)$. The generalization to a complete set of local hermitian fields is straightforward and will be left to the reader. Completeness is used here in the sense that if, for any operator T,

$$[T, \varphi(x)] = 0 \qquad \text{for all } x, \tag{2}$$

then T is a c-number. By locality is meant

$$[\varphi(x), \varphi(y)] = 0\,, \qquad \text{for } x-y \text{ space like.} \tag{3}$$

The usual assumptions are also made regarding positive metric, mass and energy states, Lorentz invariance, and the uniqueness of the vacuum state [2].

1. – The assumption (1) is the same as

$$[A(x+\xi), A(y+\xi)] = D(x-y) \tag{4}$$

for any ξ. Consider the commutator

$$[A(z), D(x-y)] = [A(z), [A(x+\xi), A(y+\xi)]]\,. \tag{5}$$

Using the Jacobi identity, this is

$$\begin{aligned}[A(z), D(x-y)] = [A(x+\xi), [A(z), A(y+\xi)]] &+ \\ + [A(y+\xi), [A(x+\xi), A(z)]]&\,. \end{aligned} \tag{6}$$

This must hold for all ξ. In particular it holds for ξ such that both $y+\xi$ and $x+\xi$ are space like with respect to z. Such ξ always exist. The locality condition then requires

$$[A(z), D(x-y)] = 0\,, \tag{7}$$

for all x, y, z. The commutator $D(x-y)$ is then a c-number, since $A(z)$ is a complete field.

2. – The concept of the generalized free field has recently been introduced by GREENBERG as a possible basis for the construction of models of interacting relativistic quantized fields. A generalized free field, $\psi(x)$, satisfies the

(2) A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956).

general axioms of quantum field theory, plus the commutation relation

$$(8)\qquad [\varphi(x), \varphi(y)] = i\int_0^\infty \mathrm{d}\mu(m)\,\Delta m(x-y)\,,$$

where $\mu(m)$ is an arbitrary measure on the interval $0 < m < \infty$. It is related to the weight function that appears in the two-point Wightman function. Here $\Delta m(x-y)$ is the commutator for an ordinary free field of mass m. It is claimed that a field, $\varphi(x)$, which satisfies (1) necessarily satisfies (8).

Let $|p, n\rangle$ be a complete set of states, each corresponding to a definite momentum p and to a set of discrete or continuous eigenvalues n. The vacuum state, denoted by $|\text{vac}\rangle$, will be supposed to be the only eigenstate of zero 4-momentum. These states are said to be complete in the sense that

$$(9)\qquad \int \mathrm{d}^4p \sum_n |p, n\rangle\langle p, n| = I\,,$$

where I is the identity operator. The sum over n is a generalized sum, involving an ordinary sum over discrete and an integral over continuous eigenvalues.

Since by the above $D(x-y)$ is a c-number function,

$$(10)\qquad D(x-y) = \langle\text{vac}|[\varphi(x), \varphi(y)]|\text{vac}\rangle =$$

$$= \int \mathrm{d}^4p \sum_n \{\langle\text{vac}|\varphi(x)|p, n\rangle\langle p, n|\varphi(y)|\text{vac}\rangle - \langle\text{vac}|\varphi(y)|p, n\rangle\langle p, n|\varphi(x)|\text{vac}\rangle\}\,.$$

By translational invariance,

$$(11)\qquad D(x-y) = 2i\int \mathrm{d}^4p \sum_n |\langle\text{vac}|\varphi(0)|p, n\rangle|^2 \sin[p\,(x-y)]\,.$$

Let

$$(12)\qquad \mathrm{d}G(p) = \sum_{\substack{n \\ p^2=-m^2}} |\langle\text{vac}|\varphi(0)|p, n\rangle|^2\,\mathrm{d}^4p\,.$$

The assumptions of positive mass and energy, and Lorentz invariance, require that

$$(13)\qquad \mathrm{d}G(p) = \int_0^\infty (2\pi)^{-3}\,\mathrm{d}\mu(m)\,\theta(p)\,\delta(p^2+m^2)\,\mathrm{d}^4p\,,$$

where $\theta(p) = 1$ if p is in the forward light cone, $= 0$ otherwise. Here $\mu(m)$

is some arbitrary measure. Now

$$(14)\qquad D(x-y)=\frac{\int \mathrm{d}\mu(m)}{(2\pi)^3}\int \mathrm{d}^4p\,\delta(p^2+m^2)\varepsilon(p)\exp[ip(x-y)]\,,$$

$$=\int_0^\infty \mathrm{d}\mu(m)i\,\Delta m(x-y)\,,$$

which proves the assertion.

3. – The measure $\mu(m)$ is in general a sum of three different types of measures,

$$(15)\qquad \mu(m)=\sum_{i=1}^{3}\mu^i(m)\,.$$

The measure of the first type gives rise to δ-functions in the integral (8),

$$(16)\qquad \mu^1(m)=\sum_{k=0}^{\infty}c_k^1\theta(m-m_h)\,,$$

where the c_k^1 are positive numbers. The measure of the second kind introduces a positive continuous function $c^2(m)$ in (8).

$$(17)\qquad \mu^2(m)=\int_0^m c^2(n)\,\mathrm{d}n\,.$$

The third type of measure, $\mu^3(m)$, is any measure not of type 1 or 2. It follows from its definition that its derivative must be almost everywhere zero, and it must assign zero weight to any single point.

A Dirac function $\delta^i(m, n)$ can be defined as a distribution relative to the measure $\mu^i(m)$ by the relation

$$(18)\qquad \int f(m)\,\delta^i(m,n)\,\mathrm{d}\mu^i(m)=f(n)\qquad \text{if}\qquad \mathrm{d}\mu^i(n)\neq 0\,,$$
$$=0\,,\qquad \text{otherwise}\,,$$

where $f(m)$ is a testing function. For measures of the first two types,

$$(19)\qquad \begin{cases}\delta^1(m_k, n_{k'})=\dfrac{\delta_{kk'}}{c_k}\,,\\[2ex] \delta^2(m,\quad n)=\dfrac{\delta(m-n)}{c^2(m)}\,,\end{cases}$$

however no such simple relation seems to exist for the third type of measure.

Consider the field

$$A(x) = \sum_{i=1}^{3} \int_0^\infty d\mu^i(m) B_m^i(x) , \tag{20}$$

where the $B_m^i(x)$ are Klein-Gordon fields of mass m,

$$(\Box - m^2) B_m^i(x) = 0 , \tag{21}$$

and satisfy the commutation relation

$$[B_m^j(x), B_n^k(k)] = i\, \delta ik\, \delta^j(m, n)\, \Delta m(x-y) . \tag{22}$$

Then it may be easily verified that $A(x)$ is a generalized free field.

$$[A(x), A(y)] = \int_0^\infty d\mu(m) i \Delta m(x-y) . \tag{23}$$

Now suppose an otherwise arbitrary generalized free field $q(x)$ is given, whose commutator is the same as in (23). It is clear that the commutator of a generalized free field completely determines all Wightman functions generated by the field. Therefore by a theorem of Wightman's [2], $q(x)$ must be unitarily equivalent to $A(x)$. Otherwise stated, any generalized free field is unitarily equivalent to a superposition of Klein-Gordon fields.

It should be noted that only the fields $B_m^1(x)$ are free fields in the ordinary sense. The fields of the second type could be considered loosely speaking as ordinary free fields with infinite renormalization constants.

Let $f_m(x)$ be a Klein-Gordon function corresponding to the mass m. A scalar product between such a function and a general field $\psi(x)$ can be defined as

$$(f_m, \psi)(t) = i \int_{x^0=t} d^3x f_m(x) \overset{\leftrightarrow}{\frac{\partial}{\partial x^0}} \psi(x) . \tag{24}$$

The asymptotic condition [3,4] is then; if the weak limit

$$\lim_{t\to\pm\infty} \langle \Phi, (f_m, \psi)(t)\Psi \rangle = \langle \Phi (f_m, \psi_{\text{in}}^{\text{out}})\Psi \rangle \tag{25}$$

(3) O. W. GREENBERG and A. S. WIGHTMAN: (unpublished).

(4) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, 1, 205 (1955).

exists, where Φ and Ψ are normalizable states, then the $\varphi_{\substack{\text{out}\\\text{in}}}$ are ordinary free fields of mass m.

Now suppose $A(x)$ is a generalized free field. By the above it has a representation (20) in terms of Klein-Gordon fields. It can readily be shown (3) that

$$\left\langle \Phi, \left(f_m \int \mathrm{d}\mu^i(n) B_n^i(x)\right)_{(t)} \Psi \right\rangle = \text{constant} + 0\left(\frac{1}{t}\right), \tag{26}$$

if $i=1$ and $\mathrm{d}\mu^1(m) \neq 0$, $=0(1/t)$, otherwise.

Thus the asymptotic fields of $A(x)$ are identical with the fields $B_m^1(x)$, moreover,

$$A_m \,\text{out}(x) = A_m \,\text{in}(x) = B_m^1(x)\,, \tag{27}$$

and there is no scattering.

It is generally assumed that the set of the in or the out asymptotic fields is complete. In the case of the generalized free field, this implies that the set of fields $B_m^1(x)$ is complete. But by (22)

$$[B_m^j(x),\, B_n^1(y)] = 0 \qquad \text{for } j=2, 3. \tag{28}$$

This implies that the fields $B_n^j(x)$ with $j=2, 3$, are null operators. This forces the conclusion that the only generalized free fields that are compatible with the existence of a complete set of asymptotic field operators are just superpositions of ordinary free fields with different masses.

RIASSUNTO (*)

Si dimostrano numerosi teoremi che si riferiscono alla possibilità di costruire un campo non-interagente con una funzione di Wightman a due punti. Essi sono: a) se $\varphi(x)$ è un campo locale completo, e $[\varphi(x), \varphi(y)] = D(x-y)$, in cui D è un operatore arbitrario che dipende da x e y solo tramite la loro differenza, allora D è funzione di un numero c; b) tali campi sono campi liberi generalizzati, secondo la definizione di Greenberg; c) ogni campo libero generalizzato è unitariamente equivalente ad una sovrapposizione di campi di Klein-Gordon, ed inoltre la condizione asintotica e l'unitarietà li riducono ad una sovrapposizione di campi ordinari con masse discrete diverse.

(*) *Traduzione a cura della Redazione.*

Functional Methods for Composite Particles.

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(ricevuto il 19 Maggio 1961)

Summary. — The aim of this paper is to show how Symanzik's functional formalism can be extended to the case of many fields of arbitrary spin, which may form composite particles of arbitrary spin and mass spectra. It will be shown that this can be done whether or not interpolating fields exist for the composite particles, and that the s-matrix is the same in either case.

1. – Preliminary definitions.

We suppose that we have a system of particles $\{\beta_r\}$, where r is a general purpose label. A subset $\{M_r\}$ of the β_r will be the system of elementary particles. The r-th particle, β_r has mass m_r, spin s_r, etc.

A wave function $\varphi_r(x)$ for β_r will be an element of a spin-isospin space V_r. We shall take the free wave-equation for β_r to be (1)

$$Q_x^r\,\varphi_r(x) \equiv (i\alpha_r^\mu\,\partial_\mu - m_r)\,\varphi_r(x) = 0\,,$$

where α_r^μ ($\mu = 1, 2, 3, 4$) are certain matrices over V_r, such that there exists a polynomial $\lambda_r(x)$ of degree $2S_r$ such that (2)

$$\lambda_r(\alpha_r p)(\alpha_r p - m_r) = m_r^2 - p^2\,.$$

We can deduce (1) the existence of a unimodular matrix Λ_r such that $\Lambda_r\alpha_r^\mu = \alpha_r^{\mu *}\Lambda_r$, and we define $\bar{\varphi}_r(x) = \varphi_r(x)^*\Lambda_r$. If D_λ denotes one of the invariant

(1) D. R. Corson shows that it is always possible to do this in his book *An Introduction to Tensors, Spinors and Relativistic Wave Equations.*

(2) See chap. II of H. Umezawa: *Quantum Field Theory*, (Amsterdam, 1956).

functions associated with the Klein-Gordon equation we put $\Delta_\alpha^{(r)} = \lambda_r(i\alpha_r\partial)D_\alpha$, so that $\Delta_\alpha^{(r)}$ is the corresponding invariant function for the full wave-equation of β_r. Each β_r is a composite of particles $M_{b_1}, \dots M_{b_n}$ and V_r is an invariant subspace of $V_{b_1} \times V_{b_2} \times \dots V_{b_n}$.

2. – Field operators.

In defining state-space and Heisenberg operators a different approach from the usual will be adopted. Firs, for each β_r, an abstract in-field operator will be defined, and these will generate a ring K. State-space will be defined as a K-module, and its metric properties will follow from the structure of K. This structure will be taken to be basic to all quantum field theories; a particular theory of interaction will be given by assigning to each M_r an element of K which will be the Heisenberg interpolating field operator. These are to carry all the information about the interaction. However, being elements of K, they are expressible as a power series (3) in the in-field operators, with certain coefficients. These coefficients, therefore, carry all the information, and they will obey a consistency condition which has been given by NISHIJIMA (4), SYMANZIK (5) and others. The out-fields will be defined in terms of the interpolating fields, and a method of Zimmermann points to the existence of an S-matrix.

For each β_r, define abstract symbols $A_r^{\text{in}}(x)$ and $A_r^{\text{in}}(x)^*$, which are to be thought of as elements of V_r and V_r^*. Take then to be generators of a ring K with the constraint:

(1) (6)
$$\begin{cases} [A_r^{\text{in}}(x), \bar{A}_s^{\text{in}}(y)]_\pm = i\Delta^{(r)}(x-y)\,\delta_{rs}\,, \\ [A_r^{\text{in}}(u), A_s^{\text{in}}(y)]_\pm = 0\,. \end{cases}$$

The $\pm$ sign follows the usual convention ($+$ when β_r and β_s belong to the same family of fermions and $-$ otherwise). From the method of construction, the centre of K is C, the field of complex numbers; $A_r^{\text{in}}(x) \to A_r^{\text{in}}(x)^*$ is to be an antilinear antiautomorphism and will be referred to as Hermitian con-

(3) By a power series is meant something of the general shape:

$$\Phi(x) = \sum \frac{1}{r!} \int \dots \int \mathrm{d}y_1 \dots \mathrm{d}y_r : \Phi_{\text{in}}(y_1) \dots \Phi_{\text{in}}(y_r) : \; C(x, y_1 \dots y_r)\,.$$

(4) K. NISHIJIMA: *Phys. Rev.*, **111**, 995 (1958).

(5) K. SYMANZIK: *Journ. Math. Phys.*, **1**, No. 4, 249 (1960).

(6) It has been assumed that, unlike the neutral scalar or Majorana particle, A and $\bar{A}$ are not linearly related. The modification necessary in this case is trivial.

jugation. Let Ω be the left K-module generated by a symbol $\rangle$, with the restriction

$$(2a)\qquad \begin{cases} \displaystyle\int A_r^{\text{in}}(x)\rangle \exp[i\varrho x]\,\mathrm{d}x = 0\,, \\ \displaystyle\int \overline{A}_r^{\text{in}}(x)\rangle \exp[i\varrho x]\,\mathrm{d}x = 0\,. \end{cases}\qquad \varrho > 0$$

Under Hermitian conjugation we get a right K-module Ω^* generated by a symbol $\langle$ satisfying

$$(2b)\qquad \begin{cases} \displaystyle\int \langle A_r^{\text{in}}(x) \exp[-i\varrho x]\,\mathrm{d}x = 0\,, \\ \displaystyle\int \langle \overline{A}_r^{\text{in}}(x) \exp[-i\varrho x]\,\mathrm{d}x = 0\,. \end{cases}\qquad \varrho < 0$$

We identify Ω with state-space, so that for example $\overline{A}_r^{\text{in}}(x)\overline{A}_s^{\text{in}}(y)\rangle$ is a state of one β_r and one β_s. By using the commutation rules (1), we can induce a mapping $\Omega^* \times \Omega \to C$ by setting $\langle 1 \rangle = 1$ ([7]). It is easily verified that this mapping is a positive definite metric on Ω.

We add the assumption

$$(3)\qquad Q_x{}^r A_r^{\text{in}}(x) = 0\,.$$

This is clearly compatible with (1), and together with (2a) and (2b) implies that $\langle A_r^{\text{in}}(x)\rangle = 0$ ([8]). For each inhomogeneous Lorentz transformation $\{a, \Lambda\}$ we can construct a unitary element $U(a, \Lambda)$ of K with $U(a, \Lambda)A_r^{\text{in}}(x)U(a, \Lambda) = A_r^{\text{in}\prime}(x')$, in the usual way. For example $U(a, 1) = \exp[ia^\mu P_\mu]$ with

$$P_\mu = \sum \frac{i}{2} \int_\sigma \overline{A}_r^{\text{in}}(x)\alpha^\nu \overleftrightarrow{\partial}_\mu A_r^{\text{in}}(x)\,\mathrm{d}\sigma_\nu\,.$$

To each M_r assign a Heisenberg operator, which is to be an element $\psi_r(x)$ of K. They must satisfy the following axioms:

I) Causality:

$$(4)\qquad [\psi_r(x), \psi_s(y)]_\pm = [\psi_r(x), \overline{\psi}_s(y)]_\pm = 0 \qquad \text{for } x \sim y.$$

([7]) Under the mapping we obtain the product of $\langle X^*$ and $Y\rangle$ by arranging X^*Y in normal order and taking $\langle X^*Y\rangle$ as the scalar part.

([8]) (2a) and (2b) imply $\int\langle A(x)\rangle \exp[i\varrho x]\,\mathrm{d}x = 0$ for $\varrho > 0$ and $\varrho < 0$. (3) implies that it vanishes also for $\varrho \sim 0$.

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II) Invariance:

$$U(a, \Lambda)\,\psi_r(x)\,U_r(a, \Lambda)^* = \psi'_r(x')\,. \tag{5}$$

III) Asymptotic condition:

Suppose β_r is a composite of $M_{b_1} \dots M_{b_n}$. Then let

$$A_r(x, \xi) = MT[(\psi_{b_1}(x+\xi_1) \dots \psi_{b_n}(x+\xi_n)]\,, \tag{6}$$

where $\xi_1 + \dots + \xi_n = 0$, M operates on the spin-isospin indices and maps $V_{b_1} \times \dots \times V_{b_n} \to V_r$, and the T-product has the usual meaning. Let

$$A_r^{\text{in}}(x, \xi) = A_r(x, \xi) - \int \Delta_{\text{ret}}^{(r)}(x - x')\,Q_{x'}^{(r)} A_r(x', \xi)\,\mathrm{d}x'\,,$$

in the sense of weak operator convergence (9). Then the Fourier transform of $A_r^{\text{in}}(x, \xi)$ is proportional to that of $A_r^{\text{in}}(x)$, and we may write

$$A_r^{\text{in}}(x) = \lim_{\xi \to 0} \frac{A_r^{\text{in}}(x, \xi)}{F_r(\xi)} \tag{7}$$

for a certain $F_r(\xi)$.

It follows from III that if an element of K commutes with every ψ, and $\overline{\psi}$, then it commutes with every A^{in} and $\overline{A}^{\text{in}}$, and hence belongs to C.

3. – Sources.

For each $\psi_r(x)$ we introduce an auxiliary quantity $J(x)$, the « source » of M_r. The J's are not to be thought of as elements of K, but as purely symbolic aids. $J_r(x)$ is to be an element of V_r and is to have the commutation rules:

$$[J_r(x), J_s(y)]_\pm = [J_r(x), \bar{J}_s(y)]_\pm = [J_r(x), \psi_s(y)]_\pm = [J_r(x), \overline{\psi}_s(y)]_\pm = 0\,. \tag{8}$$

Functional derivatives with respect to the J's will be defined in the usual way by (10)

$$\delta F = F[J + \delta J] - F[J] = \sum_r \int \left(\delta \bar{J}_r(x) \frac{\delta F}{\delta \bar{J}_r(x)} + \frac{\delta F}{\delta J_r(x)} \delta J_r(x) \right) \mathrm{d}x\,. \tag{9}$$

(9) $X = Y$ in the sense of weak operator convergence (w.o.c.) means that $\langle\alpha|X|\beta\rangle = \langle\alpha|Y|\beta\rangle$ for all $\langle\alpha|$, $|\beta\rangle$ in Ω^*, Ω.

(10) See D. V. Shirkov and N. N. Bogoliubov: *Introduction to the theory of quantized fields*, p. 566.

To exemplify the use of sources, a theorem concerning the ring K will be stated, which will be useful later. Suppose that each $A_r^{\text{in}}(x)$ of K has a source $\theta_r(x)$ with appropriate commutation properties. For any X of K define $X(\theta)=U(\theta)\,X\,U(\theta)^*$, where

$$U(\theta)=\exp\left[\sum_r\int(\bar\theta_r(x)\,A_r^{\text{in}}(x)-\bar A_r^{\text{in}}(x)\theta_r(x))\,\mathrm{d}x\right].$$

Then if σ is any spacelike surface, we have

$$(10)\qquad X=:\exp\left[\int_\sigma\sum_r(\bar A_r^{\text{in}}(x)\alpha_\mu\delta/\delta\bar\theta_r(x)-\delta/\delta\theta_r(x)\alpha_\mu A_r^{\text{in}}(x))\mathrm{d}\sigma^\mu\right]:\;X(\theta)\Big|_{\theta=0}.$$

The point of this equation is that if the vacuum expectation values of all multiple commutators of X with A_r^{in} and $\bar A_r^{\text{in}}$ vanish, then X vanishes.

For a particle $\beta_r\,(=M_{b_1}\dots M_{b_n})$ define

$$(11)\qquad \bar D_r(x)=\lim_{\xi\to0}\frac{1}{F_r(\xi)}\,M\,\frac{\delta^n}{\delta\bar J_{b_1}(x+\xi_1)\dots\delta\bar J_{b_n}(x+\xi_n)},$$

where $\xi_1+\dots+\xi_n=0$ just as in (6). Of course (11) is only meaningful in the correct circumstances. As a matter of notation, it will often be convenient to swallow the particle suffix r into the co-ordinate x. Integration over x is then extended to include summation over r. We will use the notation $\delta F/\delta\bar J(x)=F^x$ and $\delta F/\delta J(x)=F_x$ for any functional F. Suffixes at the top transform contragrediently to those at the bottom, as in tensor analysis.

If we define

$$A_r^{\text{out}}(x,\xi)=A_r(x,\xi)-\int A_{\text{av}}^{(r)}(x-x')Q_{x'}{}^r A_r(x',\xi)\,\mathrm{d}x',$$

in the sense of weak operator convergence [9], it follows from Zimmerman's work that $[A_r^{\text{out}}(x,\xi),\,A_s^{\text{out}}(y,\eta)]_\pm=[A_r^{\text{in}}(x,\xi),\,A_s^{\text{in}}(y,\eta)]_\pm$ and $[A_r^{\text{out}}(x,\xi),\bar A_s^{\text{out}}(y,\eta)]_\pm=[A_r^{\text{in}}(x,\xi),\bar A_s^{\text{in}}(y,\eta)]_\pm$ since the ψ's satisfy the conditions of his paper [11]. In particular we can define $A_r^{\text{out}}(x)=\lim_{\xi\to0}\big(1/F_r(\xi)\big)A_r^{\text{out}}(x,\xi)$. Then $A^{\text{in}}\to A^{\text{out}}$ is an automorphism of K since the commutation rules are preserved. This suggests the existence of a unitary element S of K with the properties $A_r^{\text{out}}(x)=S^*A_r^{\text{in}}(x)S$ and $S\rangle=\rangle$. This will be assumed.

([11]) W. Zimmerman: *Nuovo Cimento*, **10**, 597 (1958).

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4. – Functionals.

With the usual notation define

$$T[J] = T\exp\left[i\int\sum_r \left(\bar{J}_r(x)\psi_r(x) + \overline{\psi}_r(x)J_r(x)\right)\mathrm{d}x\right], \tag{12}$$

so that T generates the time-ordere products [12]. We have $T^*T = TT^* = 1$ From the asymptotic condition we get

$$A^{\text{out}}(x)T - TA^{\text{in}}(x) = i\int \Delta(u-y)Q_y\bar{D}(y)T\,\mathrm{d}y \qquad ({}^{13,9}), \tag{13}$$

A similar formula holds with $\bar{A}$ instead of A and $D(y)$ instead of $\bar{D}(y)$. (13) may be rewritten

$$[A^{\text{in}}(x), ST] = i\int \Delta(x-y)Q_y\bar{D}(y)T\,\mathrm{d}y\,, \tag{14}$$

Define

$$V[J] = :\exp\left[\int\sum_r \left(\bar{A}_r^{\text{in}}(x)Q_x{}^r\bar{D}_r(x) + D_r(x)\overleftarrow{Q}_x{}^r A_r^{\text{in}}(x)\right)\mathrm{d}x\right]:. \tag{15}$$

where $\overleftarrow{Q}_x{}^r = -i\alpha_r^\mu\overleftarrow{\partial}_\mu - m_r$. Then we have

$$[A^{\text{in}}(x), V] = i\int \Delta(x-y)Q_y\bar{D}(y)V\,\mathrm{d}y\,, \tag{16}$$

so that

$$[A^{\text{in}}(x), (ST - V\langle T\rangle)] = i\int \Delta(x-y)Q_y\bar{D}(y)(ST - V\langle T\rangle)\,\mathrm{d}y \qquad ({}^{9}). \tag{17}$$

Now $\langle(ST - V\langle T\rangle)\rangle = 0$ since $\langle S = \langle$ and $\langle V\rangle = 1$. Hence it follows, by iterating (17) and taking the vacuum expectation value, that the vacuum expectation value values of the multiple commutators of $ST - V\langle T\rangle$ with A and $\bar{A}$ vanish. Hence, by (10), $ST = V\langle T\rangle$. Now $T|_{J=0} = 1$, so that we have an explicit construction for S:

$$S = V\langle T\rangle|_{J=0}\,. \tag{18}$$

(12) See reference (5) and the *Proc. of the 1960 Rochester Conf.*, p. 211.
(13) For the justification of the use of $\bar{D}(y)$ in these circumstances see ref. (11).

We can now construct the asymptotic condition for Symanzik's functional

$$(19)\qquad R[K,J] = T^*\left[J+\frac{iK}{2}\right]T\left[J-\frac{iK}{2}\right] = T\left[J-\frac{iK}{2}\right]S^*ST\left[J-\frac{iK}{2}\right] =$$

$$= V\left[J+\frac{iK}{2}\right]V\left[J-\frac{iK}{2}\right]\left\langle T^*\left[J+\frac{iK}{2}\right]\right\rangle\left\langle T\left[J-\frac{iK}{2}\right]\right\rangle = V[J]\langle R\rangle \quad (^{14}),$$

where in (19) we have used the formula $e^{\alpha}e^{\beta} = e^{\alpha+\beta}e^{\frac{1}{2}[\alpha,\beta]}$ if $[\alpha,[\alpha,\beta]] = [\beta,[\alpha,\beta]] = 0$, to derive the last line. The set of functionals obeying an equation like (19) form a ring. For suppose $X = V\langle X\rangle$ and $Y = V\langle Y\rangle$, then $X+Y = V\langle(X+Y)\rangle$ and $XY = V\langle XY\rangle$ where $\langle XY\rangle = \langle X\rangle \overleftrightarrow{\Sigma}\langle Y\rangle$ and $\overleftrightarrow{\Sigma} = \langle\overleftarrow{V}\overrightarrow{V}\rangle$ is the functional differential operator

$$(20)\qquad \exp\left[\sum_r\iint \overleftarrow{D}_r(x)\overleftarrow{Q}_x^{\,r}\, i\, \Delta_r^+(x-y)\overrightarrow{Q}_y^{\,r}\,\overrightarrow{D}_r(y)\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}\right] \quad (^{15}).$$

Symanzik's paper on Green's functions extends itself in a natural way to the general case with the following trivial alterations. i) The suffixes on functionals are no longer symmetric; when they refer to the same boson they are symmetric, and to the same fermion antisymmetric. Also we must distinguish between covariant and contravariant suffixes. ii) the Δ-functions in his paper generalize to the ones defined here, but this alters none of the conclusions concerning the singularities. iii) The operator V and $\overleftrightarrow{\Sigma}$ extend to the definitions (15) and (20). iv) In appropriate places commutators must be redlaced by anticommutators. *E.g.* we can derive for R the equations:

$$(21)\qquad R_{x,y} \pm R_{y,x} = i[R_x, R_y]_\pm\,, \qquad R_x^{\;y} \pm R_x^{\;y} = i[R_x, R^y]_\pm \qquad (^{16}),\ \text{etc.}$$

From (19) we obtain $\psi(x) = R^x|_{J=0} = V\langle R^x\rangle|_{J=0}$ which is an expansion of $\psi(x)$ in terms of the in-field operators. From the nature of R we known that $R_{x,y} = 0$ unless $x > y$, so that (21) ensures axiom I. Axiom II follows from the general invariance. This explains the remark above, and eq. (21), which, together with retardedness, define R, are the consistency conditions on the coefficients of the expansion of $\psi(x)$ in terms of $A^{\text{in}}(x)$ and $\bar{A}^{\text{in}}(x)$.

(14) See eq. (14) of reference (5).
(15) See the bottom of page 263 of reference (5).
(16) See eq. (18) of reference (5).

5. – The S-matrix.

If a composite particle has an interpolating field operator, the diagram

$$\begin{array}{ccc} A^{\text{out}}(x,\xi) \leftarrow & A(x,\xi) & \rightarrow A^{\text{in}}(x,\xi) \\ \downarrow & & \downarrow \\ A^{\text{out}}(x) & & A^{\text{in}}(x) \end{array} \tag{22a}$$

is completed to form the commutative diagram

$$\begin{array}{ccccc} A^{\text{out}}(x,\xi) & \leftarrow & A(x,\xi) & \rightarrow & A^{\text{in}}(x,\xi) \\ \downarrow & & \downarrow & & \downarrow \\ A^{\text{out}}(x) & \leftarrow & A(x) & \rightarrow & A^{\text{in}}(x) \end{array} \tag{22b}$$

The vertical arrows all denote the linear maps which destroy the ξ dependence, and the horizontal arrows denote the well-known (w.o.c.) mappings. The functionals T and R derived above were constructed only from the Heisenberg fields $\psi(x)$ of the particles M_r. Supposing that some of the composite particles β_s have interpolating fields $A_s(x)$, we can ascribe source $U_r(x)$ to them and can construct a new functional:

$$T' = T \exp\left[i\int\left\{\sum_r \left(\bar{J}_r(x)\psi_r(x) + \overline{\psi}_r(x)J_r(x)\right) + \right.\right.$$
$$\left.\left. + \left(\overline{A}_s(x)U_s(x) + \overline{U}_s(x)A_s(x)\right)\right\}\mathrm{d}x\right], \tag{23}$$

and a new operators V' obtained by replacing $\overline{D}_s(x)$ by $\delta/\delta\overline{U}_s(x)$ in V. The quantities T' and V' correspond to the case where β_s has been treated as elementary. T and V correspond to the case where β_s is treated as composite. In the «elementary picture» we obtain an S-matrix $S'=V'\langle T'\rangle|_{J=U=0}$, and in the «composite picture» an S-matrix $S=V\langle T\rangle|_{J=0}$. We assert that $S=S'$, so that as far as scattering is concerned there is no test to discriminate between compositeness or elementarity. To see this, note that the in-field operators are the same in both cases, so that we are only interested in proving that the coefficients in the expansions of S and S' coincide on the mass-shell. That is to say, we wish to show that

$$\int\bar{f}(x)Q_x\, T(A(x)\dots)\,\mathrm{d}x = \lim_{\xi\to 0}\frac{1}{F(\xi)}\int\bar{f}(x)Q_x\, T(\psi_{b_1}(x+\xi_1)\dots\psi_{b_n}(x+\xi_n)\dots)\,\mathrm{d}x\,,$$

where $f(x)$ is any function such that $Q_x f(x)=0$. Using Zimmerman's argu-

ments (11) we can show that the left-hand side is:

$$i\int_\sigma \bar{f}(x)\alpha^\mu\big(\langle A^{\text{out}}(x)\,T(\ldots)\rangle - \langle T(\ldots)\,A^{\text{in}}(x)\rangle\big)\,\mathrm{d}\sigma_\mu\,,$$

for any spacelike surface σ, and similarly the right-hand side is

$$i\int_\sigma \bar{f}(x)\alpha^\mu \lim_{\xi\to 0}\frac{1}{F(\xi)}\big(\langle A^{\text{out}}(x,\xi)\,T(\ldots)\rangle - \langle T(\ldots)\,A^{\text{in}}(x,\xi)\rangle\big)\,\mathrm{d}\sigma_\mu\,.$$

The two expressions are the same. Repeating the argument for each term in each coefficient proves the assertion that $S = S'$. Essentially, we have gone round diagram (22*b*) in two different ways and have arrived at the same result.

* * *

I am grateful to Dr. J. C. POLKINGHORNE for advice and encouragement and to D.S.I.R. for a grant.

RIASSUNTO (*)

Scopo di questo lavoro è di mostrare come il formalismo funzionale di Symanzik possa essere esteso al caso di molti campi di spin arbitrario, che possono formare particelle composite con spin e spettri di massa arbitrari. Si dimostrerà che ciò può essere fatto sia che esistano o no campi interpolanti per le particelle composte, e che la matrice s è la stessa in entrambi i casi.

(*) *Traduzione a cura della Redazione.*

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Further Measurements of the Size-Spectrum of Extensive Air Showers (*).

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Summary. — Measurements on the size-spectrum of extensive air showers have been made over a period of about one year to include approximately 10^5 events. It is found that errors of about 5% in determining the number of particles traversing the scintillator make it impossible to observe time variations of less than 8%. For showers containing a total number of particles $10^5 < N < 10^7$, the spectrum for vertical incidence is $K_v(\geqslant N) = (2.26 \pm 0.14) \cdot 10^{-7}(N/10^6)^{-(1.540\pm 0.006)}\ \mathrm{s^{-1}m^{-2}sr^{-1}}$.

1. – Introduction.

Measurements on the integral size-spectrum of the extensive air showers of the cosmic radiation described in an earlier paper (1) have been continued over a period of about one year. Briefly, the method employed consists in finding the integral spectrum of the pulse-size distribution occurring in a large liquid scintillator. It has been shown (2) that if the experimental spectrum of particles traversing a single scintillator be found in the form

$$Q(\geqslant p) = Q_0 p^{-\gamma}\ \mathrm{s}^{-1}\,, \tag{1}$$

where p is the number of particles intercepted by the scintillator, then it is possible to compute the constants in the integral spectrum for the rate of

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(1) J. R. Green and J. R. Barcus: *Nuovo Cimento*, **14**, 1356 (1959).

(2) J. R. Green: *Nuovo Cimento*, **14**, 1342 (1959).

occurrence of the extensive air showers as a function of size for a functional form

$$K(\geqslant N) = K_v \cos^m \theta\, N^{-\gamma}\ \text{s}^{-1}\,\text{m}^{-2}\,\text{sr}^{-1}, \tag{2}$$

where m is a function of altitude and is taken to be 6.5 for the elevation of Albuquerque (1575 m.s.l.), where N is the total number of particles in the shower, and where γ is the same exponent that occurs in (1).

2. – Results.

During the period from December 1959 to October 1960 ten separate runs were made, each run lasting for about two weeks and comprising about 10^4 events. Table I gives the time of each run, the total number of events, and the required minimum number of particles in the scintillator.

TABLE I. – *General data for runs 1959-1960.*

Run	Period	Time (min)	Number of events	Minimum number in upper scint.
1-59	12/10/59 - 12/21/59	14913	7271	185
2-59	12/23/59 - 1/ 4/60	16497	9029	185
1-60	1/12/60 - 1/24/60	15980	8669	185
2-60	2/13/60 - 3/ 4/60	27517	11369	241
3-60	3/22/60 - 4/14/60	31051	11491	250
4-60	4/29/60 - 5/13/60	19522	11040	185
5-60	5/ 1/60 - 6/16/60	20349	10777	195
6-60	6/29/60 - 7/15/60	22980	11100	195
7-60	7/25/60 - 8/11/60	22774	8670	242
8-60	9/20/60 - 10/10/60	28350	11043	231

In the analysis of each of the runs, seven divisions were made according to the size p in the scintillator for the purposes of obtaining the integral size spectrum. A finer division was felt to be unnecessary in view of the earlier detailed analysis of the spectrum. A typical integral spectrum so obtained is shown in Fig. 1; the experimental errors are about the size of the points shown. The individual spectra could thus be consistently fitted on a log-log plot by a straight line (neglecting the fall-off caused by saturation of the photomultiplier for numbers of particles in excess of a few times 10^3); this indicates the correctness of the assumption of a power law for the distribution. Values of the (negative) logarithmic slope obtained in this way are given in

Table II. The error in the logarithmic slope was estimated by finding the possible straight lines that could reasonably be drawn through the data points. If an average of these values for the logarithmic slope be taken and the error found by comparison of these values with the average, the following results are obtained: average value 1.540, standard deviation of the average 0.006, standard deviation of a datum 0.018. This indicates that the errors assigned in Table II probably represent an overestimation of the actual error of the individual slopes.

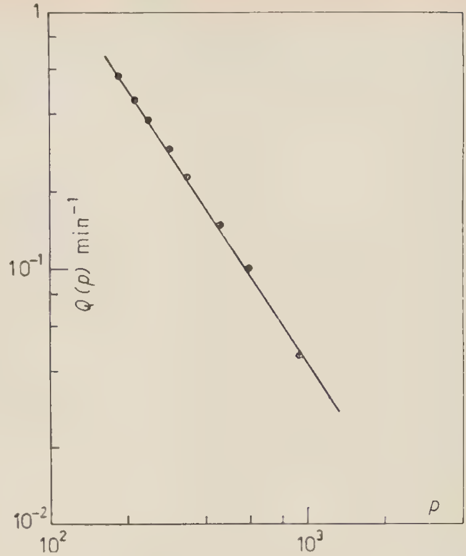

Fig. 1. – Typical integral counting rate (run 4-60). The errors in the rates are approximately the size of the points.

Before proceeding to the determination of the constant of the distribution, it is necessary to show that there is an important error associated with the determination of p, the number of particles traversing the scintillator. In Fig. 2 will be found the average integral counting rates for 300 or more particles and for 800 or more particles for each of the ten runs plotted for the times over which the runs extended. On the basis of the statistical error due to the number of counts alone, the average rate for 300 or more particles should be (0.266 ± 0.001) min^{-1} and that for 800 or more particles should be (0.0585 ± 0.0005) min^{-1}. When the individual data are compared with the average values, however, the following results are obtained: for 300 or more particles, average rate; (0.266 ± 0.007) min^{-1} with the standard deviation of a datum of 0.020 min^{-1}

TABLE II. – *Values of the logarithmic slope with estimated errors.*

Run	Slope	Run	Slope
1-59	1.541 ± 0.736	4-60	1.524 ± 0.027
2-59	1.540 ± 0.028	0-60	1.567 ± 0.027
1-60	1.535 ± 0.035	6-60	1.515 ± 0.020
2-60	1.518 ± 0.030	7-60	1.555 ± 0.028
3-36	1.567 ± 0.025	8-60	1.537 ± 0.025

for 800 or more particles, average rate $(0.0585 \pm 0.0015)\ \text{min}^{-1}$ with the standard deviation of a datum of $0.0044\ \text{min}^{-1}$. Thus the actual errors are considerably larger than those predicted on the basis of the statistics alone.

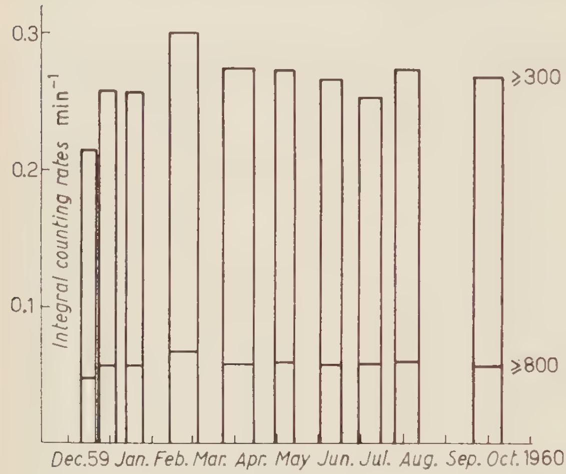

Fig. 2. – Integral counting rates for 300 or more particles and for 800 or more particles in the scintillator as a function of the time during which the measurements were made.

If now it be assumed that the true rate of events during the entire period of measurement was given by the expression (1) and that the constant term Q_0 and the logarithmic slope were indeed constant over the entire period, then the fractional error in the determination of the number of particles traversing the scintillator is related to the fractional deviation in the observed rates through the relation

$$\text{(3)} \quad \mathrm{d}p/p = (1/\gamma)(\mathrm{d}Q/Q)\,.$$

Using the average value of 1.540 for γ and the deviation of an individual datum, we obtain the following fractional errors in the determination of the number of particles: 300 or more particles, 5.0%; 800 or more particles, 4.9%. The consistency of the conclusions drawn from the assumptions and the reasonable value of the fractional error in the number of particles lead one to believe that the observed variations in the rates can adequately be explained by the single assumption of an uncertainty of 5% in determining the number of particles traversing the scintillator. A further confirmation is obtained if one examines the rates and the slopes as a function of time as displayed in Fig. 2

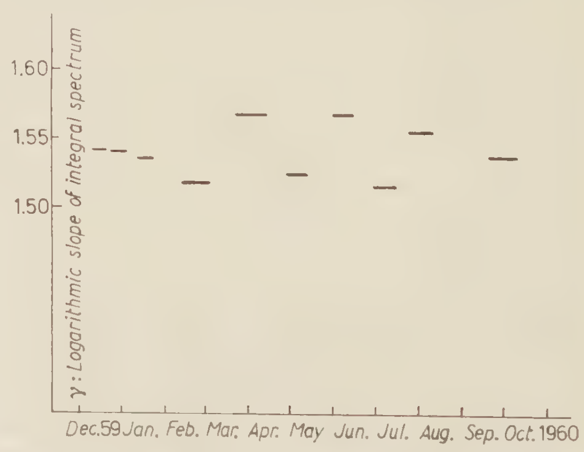

Fig. 3. – Values of the logarithmic slope for the integral counting rate curves as a function of the time during which the measurements were made.

and 3; the deviations are seen to occur in an entirely random fashion. This then indicates that if there are any systematic variations in the rates during the year, they must be smaller than the variations caused by the error in determining the number of particles in the scintillator. Thus any seasonal variation in the rate of occurrence of the extensive air showers must be less than about 8%. Investigations of the diurnal variations in the rates of extensive air showers as measured by arrays of Geiger-Müller counters (3) have shown that variations due to atmospheric effects are of the order of a few tenths of a percent. This shows then the unsuitability of the single scintillator method for investigating the time variations in the extensive air showers. In general one would conclude that the G-M arrays with their on-off characteristics, stability and efficiency are more suitable for finding accurately time variations in the rate of occurrence of showers of a fixed size; whereas, the scintillators are more suitable for finding the dependency of the rate of the showers on shower size.

Values of the constant term Q_0 can be computed for each of the runs using the observed rates and the calculated slopes. On the basis of these values, one finds that the average value with its deviation is $(1.72 \pm 0.07)\cdot 10^3\ \text{min}^{-1}$, and that the deviation of an individual datum is $0.21\cdot 10^3\ \text{min}^{-1}$. On the other hand, on the basis of the propagation of errors, the deviation of an individual datum should be given by

$$(4)\qquad \mathrm{d}Q_{01} = Q_0[(\mathrm{d}C/C)^2 + (\gamma\,\mathrm{d}p/p)^2 + (\ln p\,\mathrm{d}\gamma)^2]^{\frac{1}{2}}\,,$$

where C is the number of events with p or more particles traversing the scintillator for an individual run. For random counting errors, we should expect $\mathrm{d}C = C^{\frac{1}{2}}$, so that the error due to this cause is negligible compared with the other errors. Taking $\mathrm{d}p/p = 0.05$, $p = 300$, $\gamma = 1.54$, and $\mathrm{d}\gamma = 0.018$, we obtain $\mathrm{d}Q_{01} = 0.22\cdot 10^3\ \text{min}^{-1}$ in agreement with the estimate obtained above.

The rate of events with p or more particles traversing the scintillator is thus given by

$$(5)\qquad Q(\geqslant p) = (1.72 \pm 0.07)\cdot 10^3 p^{-(1.540\pm 0.006)}\ \text{min}^{-1}\,.$$

If the theory of reference (2) is applied, the integral spectrum of the extensive air showers at Albuquerque (1575 m.s.l.) is found to be

$$(6)\qquad K_v(\geqslant N) = (2.26 \pm 0.14)\cdot 10^{-7}(N/10^6)^{-(1.540\pm 0.006)}\ \text{s}^{-1}\,\text{m}^{-2}\,\text{sr}^{-1}$$

for vertical incidence.

(3) J. Daudin, P. Auger, A. Cachon and A. Daudin: *Nuovo Cimento*, **3**, 1017 (1956).

3. – Discussion.

The earlier results obtained for the number spectrum are thus confirmed with a considerable increase in accuracy. Because of the broad response of the single scintillator, it is difficult to say exactly for what range of shower sizes this spectrum is applicable. The problem can, however, be approached in the following way. The rate at which showers of size N to $N+\mathrm{d}N$ occur at distances x to $x+\mathrm{d}x$ (where distances are measured in terms of the characteristic shower-length r_2) is

$$R(N, x)\,\mathrm{d}x\,\mathrm{d}N = 2\pi r_2^2 x\,\mathrm{d}x\,BN^{-\gamma-1}\,\mathrm{d}N\,. \tag{7}$$

If there are to be p-particles in the scintillator whose area is A, there is then a functional relation between N, x, and p, namely,

$$p = AN\,f(x)/r_2^2\,, \tag{8}$$

where $f(x)$ is the normalized fractional lateral distribution function for the particles of the shower. For a fixed N to obtain p or more particles in the scintillator, the values of x must run from zero to a maximum value that is given by the inverse of function (8), that is,

$$x = x(AN/pr_2^2)\,. \tag{9}$$

Therefore, the rate of occurrence of p or more particles in the scintillator due to showers of size N to $N+\mathrm{d}N$ is obtained by integrating (7) over x from 0 to the maximum value, thus

$$q(N, p)\,\mathrm{d}N = \pi r_2^2 x^2 BN^{-\gamma-1}\,\mathrm{d}N\,, \tag{10}$$

where x is given by (9). If (10) be divided by $Q(>p)$, the result will be the normalized probability that an event with p or more particles was caused by a shower of size N to $N+\mathrm{d}N$. It turns out that the complete dependence on N and on p can be expressed in terms of the single variable $z=(AN)/(pr_2^2)$, so that the final result is

$$w(z)\,\mathrm{d}z = \pi r_2^2 (A/r_2^2)^{\gamma} g_2(\gamma, s) x^2 z^{-\gamma-1}\,\mathrm{d}z\,, \tag{11}$$

where $g_2(\gamma, s) = B/Q_0$ and is given in reference [2]. The distribution $w(z)$ is given in Fig. 4 for the Greisen approximation to the Nishimura-Kamata lateral

distribution function with $s = 1.3$ (4). This curve is used to find the most probable value for z, and hence the N for a given p. More instructive from the point of view of the contribution of the showers to the counting rate is the distribution

$$w(y)\,\mathrm{d}y = \pi r_2^2 (A/r_2^2)^\gamma g_2(\gamma, s) x^2 \exp[-\gamma y]\,\mathrm{d}y\,, \tag{12}$$

where $y = \ln z$. This distribution, which is also plotted in Fig. 4 can be used to find the range in z that must be covered to obtain a given fraction of the total counting rate. The cut-off indicated is that caused by the finite size of the scintillator tank. Table III gives the most

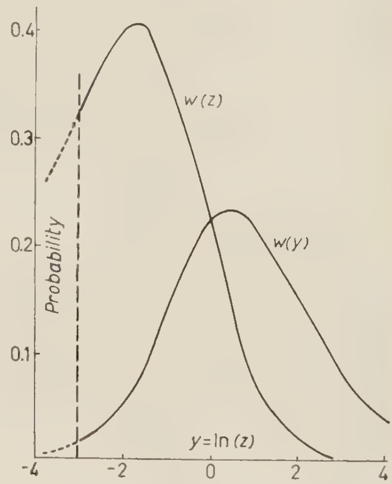

Fig. 4. – Normalized probability distribution functions $w(z)$ and $w(y)$, where $y = \ln z$, that give the probability that an event in which p or more particles are incident on the scintillator is caused by a shower of size N to $N + \mathrm{d}N$ in terms of the general parameter $z = (AN)/(pr_2^2)$.

TABLE III. – *Values of N corresponding to p or more particles in the scintillator.*

	$p = 200$ or more	$p = 1000$ or more
Most probable N	$4 \cdot 10^4$	$2 \cdot 10^5$
N for which 20% of counting rate is achieved	10^5	$5 \cdot 10^5$
N for which 80% of counting rate is achieved	$2 \cdot 10^6$	10^7
Average value of N	10^5	$5 \cdot 10^7$

probable value of N, the value of N (starting from $N_{\min}$) for which 20% of the counting rate has been achieved, the value of N for which 80% of the counting rate has been achieved, and the average value of N as obtained in reference (2) for showers producing p or more particles for the largest and smallest values measured in this experiment. In view of these values, it would seem that spectrum (6) is valid for showers $10^5 < N < 10^7$.

(4) K. GREISEN: *Progress in Cosmic Ray Physics*, vol. **3** (Amsterdam, 1956), chap. 1.

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RIASSUNTO (*)

Abbiamo eseguito per un periodo di circa un anno delle misure sullo spettro dimensionale degli sciami estesi dell'aria fino a comprendere circa 10^5 eventi. Abbiamo trovato che errori di circa il 5% nella determinazione del numero di particelle che traversano lo scintillatore rendono impossibile di osservare variazioni temporali inferiori all'8%. Per gli sciami contenenti un totale di particelle $10^5 < N < 10^7$, lo spettro per l'incidenza verticale è $K_v(\geqslant N) = (2.26 \pm 0.14) \cdot 10^{-7} \cdot (N/10^6)^{-(1.540 \pm 0,006)}\ \mathrm{s}^{-1}\,\mathrm{m}^{-2}\,\mathrm{sr}^{-1}$.

(*) *Traduzione a cura della Redazione.*

The Influence of Isotopic Composition on the Maximum in the Cosmic Ray Energy Spectra (*).

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(ricevuto il 15 Maggio 1961)

One of the interesting results obtained from measurements on the quiescent primary cosmic ray beam over the past several years is the observation that the differential rigidity spectrum of the doubly charged component is similar (differing by a scale factor) to that of the singly charged component (1,2); this has led to the belief that the spectral form is best characterized in terms of magnetic rigidity. Additionally, the observation of a maximum at low energies (1,9) has been attributed as being due to a magnetic mechanism (10) whose intensity is associated with the solar cycle. This maximum has alternately been suggested as resulting from the effect of ionization loss (11). In this note, we would like to examine the possibility that this maximum may arise in part from a misinterpretation of the data due to oversimplifying assumptions about the composition of the beam.

It has recently been observed, at least for low values of magnetic rigidity (or energy per nucleon), that the doubly charged component of cosmic rays (He-nuclei) contains an appreciable

$$[{}^3\text{He}/({}^3\text{He} + {}^4\text{He}) \sim 35\%]\,,$$

abundance of ^{3}He-nuclei (12,13). We examine here the influence of this composition on the usual interpretation of the measurements which do not recognize it, and show that this leads, in part, to a maximum in the spectrum as conventionally deduced from observations.

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(**) On leave of absence from Tata Institute of Fundamental Research, Bombay, India.

(1) F. B. MACDONALD and W. R. WEBBER: *Phys. Rev.*, **115**, 194 (1959).

(2) F. B. MACDONALD: *Phys. Rev.*, **116**, 462 (1959).

(3) P. H. FOWLER and C. J. WADDINGTON: *Phil. Mag.*, **1**, 637 (1956).

(4) P. H. FOWLER, P. S. FREIER and E. P. NEY: *Suppl. Nuovo Cimento*, **8**, 492 (1958).

(5) P. S. FREIER, E. P. NEY, J. E. NAUGLE and G. W. ANDERSON: *Phys. Rev.*, **79**, 206 (1950).

(6) H. AIZU, Y. FUJIMOTO, S. HASEGAWA, M. KOSHIBA, I. MITO, J. NISHIMURA, K. YOKOI and M. SCHEIN: *Phys. Rev.*, **116**, 436 (1959) and *Proc. Moscow Cosmic Ray Conf.*, vol. **3** (1959)

(7) K. YOKOI: private communication.

(8) A. ENGLER, M. F. KAPLON, A. KERNAN, J. KLARMANN, C. E. FICHTEL and M. W. FRIEDLANDER: submitted to *Nuovo Cimento*.

(9) A. ENGLER, F. FOSTER, T. L. GREEN and J. MULVEY: private communication.

(10) S. F. SINGER: *Suppl. Nuovo Cimento*, **8**, 342 (1958).

(11) M. V. K. APPA RAO and M. F. KAPLON: to be published.

(12) M. V. K. APPA RAO: to appear in *Phys. Rev.*

(13) M. V. K. APPA RAO: private communication.

We show first that a maximum (if it did not exist naturally) or the enhancement of one (if it existed in the natural spectrum) occurs due to plotting spectra on a basis of energy per nucleon instead of rigidity. We assume that the cosmic ray spectra at balloon altitudes will have a geomagnetic cut-off defined in terms of rigidity which is the same for all particles. Next we show, and this is the essential aspect, that a maximum arises due to the lack of recognition of the ^{3}He component in the usual measurements coupled with the assumption that these are ^{4}He.

For simplicity we assume an incident spectrum falling exponentially as a function of rigidity, incident on the detector; this spectrum consists of ^{4}He and ^{3}He nuclei, the intensity of ^{3}He with respect to ^{4}He per unit interval of rigidity being 50%; these are represented by the curves labelled ^{4}He and ^{3}He in Fig. 1. If these spectra are now expressed in terms of energy per nucleon, the ^{3}He is shifted towards a higher energy per nucleon than a ^{4}He of the same rigidity. Thus if energy per nucleon is measured, the assumption that all the nuclei are ^{4}He results in the adding of the two spectra, one shifted with respect to the other and leads to a maximum. This resultant re-expressed as a rigidity plot is shown as curve 1 in Fig. 1. Making similar assumptions as to composition on an incident spectrum with a natural maximum (Fig. 2), and performing the same operation as above leads to a large enhancement of the maximum (curve 2 in Fig. 2).

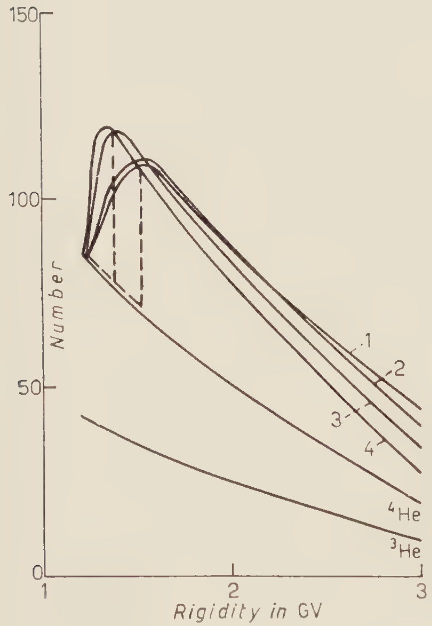

Fig. 1. – The effect of isotopic composition on measurements of energy spectra of helium nuclei. The curves ^{4}He and ^{3}He refer to exponentially falling rigidity natural spectra with the same cut-off. Curves 1, 2, 3, 4 refer to spectra as will be obtained by measurements of energy per nucleon, velocity, range and scattering without taking cognizance of the mass of the helium nucleus.

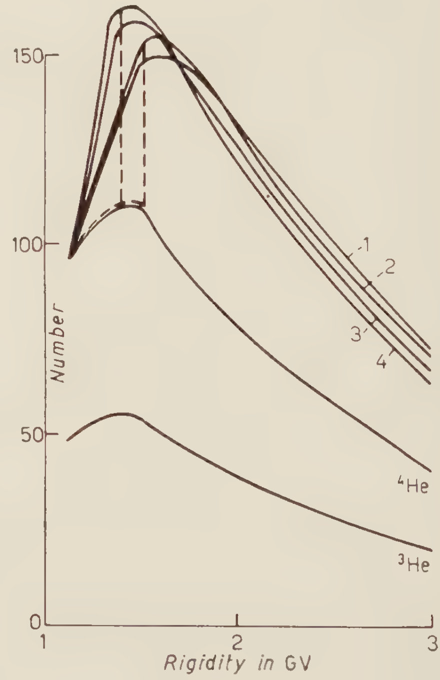

Fig. 2. – Same as Fig. 1, except the natural rigidity spectrum is assumed to have a maximum.

We now examine the effect of the assumption that all the He nuclei are ^{4}He

on the usual method of obtaining energy spectra. Differential energy spectra have been measured using nuclear emulsions or various kinds of counters. With nuclear emulsions one can measure multiple Coulomb scattering, ionization and range (R). These yield respectively a measure of $p\beta$ ($\beta = v/c$), β (assuming Z is known) and E (if the mass is known). One now transfers the rigidity spectra of the ^{4}He-^{3}He mixture to a basis of $p\beta$, β or R as the case may be and again the ^{3}He move to a higher value of the parameter for a given rigidity than the ^{4}He. If the observer does not recognize the existence of the two different masses he interprets his measurements as the sum of these two curves (one shifted with respect to the other) and obtains a maximum. This maximum persists when re-expressed in terms of rigidity and are shown respectively for $p\beta$, β and R as curves of Fig. 1. A similar handling for an incident spectrum with a natural maximum is displayed as curves 1, 3, 4 in Fig. 2.

For the case of counters to date one has usually had a measure of β and Z, the detector not being mass-sensitive. The discussion is the same as that above and similar results hold. Thus, the occurrence of a maximum in the He differential energy spectrum could be a consequence of the lack of recognition of the isotopic composition. However, a comparison of the effect of lack of isotopic recognition with the shape of the observed maxima indicate that it cannot be wholly due to the isotopic structure of the He beam.

A similar explanation may be proposed for the maximum obtained for the singly charged (proton) beam. In this case occurrence of deuterons and tritons to the extent of as little as 5% can produce a large dip on the low energy side of the maximum. This is a consequence of a relative shift for deuteron and triton curves with respect to protons opposite to that in the ^{3}He-^{4}He discussion. This effect is shown in Fig. 3 in which the natural composition is taken as 5% ^{3}H with respect to protons, all other assumptions being the same as for the ^{3}He-^{4}He discussion; the natural curves are shown for P and ^{3}H labelled

Fig. 3. – The effect of isotopic composition on the energy spectrum of singly charged particles. The assumed rigidity spectrum for protons is curve 1; curve 2 represents the spectrum for tritons, their abundance being assumed to be 5% per unit rigidity interval. Curve 3 shows the spectrum that will be obtained from a measurement of velocity without taking into account the mass of the particle.

1 and 2 and the resultant rigidity spectrum obtained from a velocity measurement without ^3H recognition is curve 3. It seems reasonable in addition that instrumental resolution would broaden these so that they would appear much smoother. One can of course also visualize an isotope effect in the case of heavy nuclei but it would not yield anything experimentally recognizable since $\Delta(Z/A)/(Z/A) \ll 1$. One should also note that for the singly charged component, since the shift is to the left, a lack of recognition of isotopic composition could result in an apparent violation of the geomagnetic cut-off; such would not be the case for the doubly charged component.

It is clear that our discussion has been rather elementary; our desire has been principally to point out the importance that lack of recognition of isotopic composition may have. Clearly this importance is quite strongly dependent on the exact measurement made and on the cut-off rigidity. For instance as the cut-off rigidity increases and $\beta \to 1$, $p\beta \to p$ and thus a measurement $p\beta$ for a given Z is to all practical purposes a measurement of rigidity and the importance of isotopic composition for spectral shape is greatly diminished. This does serve to emphasize however that it is in the non-relativistic region, where such shape is observed, that isotopic composition may be important and in any attempt at a detailed understanding it should not be ignored. For instance in the example given for the « proton » component, the low energy portion of the spectrum far beyond the left of the maximum would probably not be observed due to instrumental limitations. In this particular example, the relative amount of ^3H was chosen on the assumption (neither verified or unverified as yet) that the ^3He observed is in equilibrium with ^3H, both resulting from stripping reactions in our local system. Though the result may be suggestive, it is clearly not at all conclusive and such a hypothesis is best tested by a direct measurement; such a measurement is feasible and we plan to carry it out in the future.

On the Neutron-Proton Mass Difference.

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(ricevuto il 10 Giugno 1961)

Recent experiments on high energy electron scattering (1) have shown that the electromagnetic form factors of the proton can no longer be considered as identical since, while $F_{2p}(q^2)$ approaches zero at a value of the momentum transfer of about $25f^{-2}$, $F_{1p}(q^2)$ remains finite. At the same time on the basis of new measurements of the inelastic cross section of the deuteron, Herman and Hofstadter (2) have deduced the neutron form factors. Thus our present knowledge of the electromagnetic structure of nucleons is contained in the four form factors (3)

$$(1)\qquad \left\{\begin{aligned} F_{1p}(q^2) &= 0.12 + \frac{0.28}{1+0.214q^2} + \frac{0.60}{1+0.10q^2}\,, \\ F_{2p}(q^2) &= -0.34 + \frac{0.10}{1+0.214q^2} + \frac{1.24}{1+0.10q^2}\,, \\ F_{1n}(q^2) &= 0.32 + \frac{0.28}{1+0.214q^2} - \frac{0.60}{1+0.10q^2}\,, \\ F_{2n}(q^2) &= -0.068 - \frac{0.094}{1+0.214q^2} + \frac{1.16}{1+0.10q^2}\,. \end{aligned}\right.$$

In the light of these new results we re-evaluate here the neutron-proton mass difference.

As known, many calculations have been attempted in the past to explain the nucleon mass difference ΔM (4). In recent times Feynman and Speisman (5),

(1) R. Hofstadter, F. Bumiller and M. Croissiaux: *Proc. of the 1960 Annual Intern. Conf. on High-Energy Physics at Rochester*, p. 762; F. Bumiller, M. Croissiaux and R. Hofstadter: *Phys. Rev. Lett.*, **5**, 261 (1960) and **5**, 263 (1960); K. Berkelman, J. M. Cassels, D. N. Olson and R. R. Wilson: *Proc. 1960 Rochester Conf.*, p. 757 and *Nature*, **188**, 94 (1960).

(2) R. Herman and R. Hofstadter: *Proc. of the 1960 Rochester Conf.*, p. 767; R. Hofstadter, C. de Vries and R. Herman: *Phys. Rev. Lett.*, **6**, 290 (1961).

(3) R. Hofstadter and R. Herman: *Phys. Rev. Lett.*, **6**, 293 (1961).

(4) See for a review: S. S. Schweber, H. A. Bethe and F. de Hoffmann: *Mesons and Fields*, (vol. **1**, 1956).

(5) R. P. Feynman and G. Speisman: *Phys. Rev.*, **94**, 500 (1954 .

using a relativistic cut-off procedure based on the introduction of a fundamental length into the divergent integrals of the second-order electromagnetic self-energy, have succeeded in getting the right mass difference which, however, turns out to be a function of a cut-off parameter. Later, WICK (6) has re-calculated ΔM making use of Low's formalism (7) and of the nucleon form factors

$$(2) \qquad \begin{cases} F_{1p}(q^2) = F_{2p}(q^2) = F_{2n}(q^2) = \dfrac{1}{(1+0.0533q^2)^2}\,, \\ F_{1n}(q^2) = 0\,, \end{cases}$$

which HOFSTADTER (8) had determined from the experimental data then available. The result of this calculation is negative since the proton turns out heavier than the neutron, but it has the merit of showing that a finite mass difference can be obtained even in the frame of a local theory. Finally, CINI *et al.* (9) have used the methods of dispersion relations to determine ΔM. Their result, based on the form factors (2) gives again a proton heavier than the neutron.

The inproved knowledge of the neutron structure now requires a new evaluation of ΔM.

Our calculation has been performed by using the second order perturbation in e. There seems in fact to be no modification if we use perturbation theory instead of a dispersion technique for this problem. Following standard methods we obtain for the self energy of the nucleon

$$\Delta E = \frac{e^2}{(2\pi)^4 i}\,\bar{V}(P)\int\frac{\mathrm{d}^4 q}{q^2}\left[\gamma_\mu F_1(q^2) + \frac{i\mu}{4M}[\gamma_\mu, \boldsymbol{q}]F_2(q^2)\right]\frac{-i(\boldsymbol{p}-\boldsymbol{q})+M}{(p-q)^2+M^2}\cdot \\ \cdot\left[\gamma_\mu F_1(q^2) - \frac{i\mu}{4M}[\gamma_\mu, \boldsymbol{q}]F_2(q^2)\right]V(p)\,,$$

where F_1 and F_2 are now the form factors (1).

Unfortunately, the presence of the constants in (1) prevents a complete cancellation of the divergences, so that the mass difference is again a function of a cut-off parameter. There is, however, no real need of any cut-off because the divergences may be ignored. In fact when $q^2=0$ we obtain a mass difference

$$\Delta M = 5.685m\,,$$

where m is the electron mass, which is approximately 2.2 times bigger than the experimental value $2.52m$; ΔM increases steadily with the value of the cut-off. For $q^2 = 5f^{-2}$ we get $\Delta M = 5.740m$ and for $q^2 = 10f^{-2}$, $\Delta M = 5.850m$.

If we represent the mass difference as

$$\Delta M = A_0 + A_1(\mu) + A_2(\mu^2)\,,$$

(6) G. C. WICK: *Proc. of the Seventh Annual Rochester Conf.* (1957).
(7) F. LOW: *Phys. Rev.*, **97**, 1922 (1955).
(8) R. HOFSTADTER, F. BUMILLER and M. R. YEARIAN: *Rev. Mod. Phys.*, **30**, 482 (1958).
(9) M. CINI, E. FERRARI and R. GATTO: *Phys. Rev. Lett.*, **2**, 7 (1959).

where $A_1(\mu)$ and $A_2(\mu^2)$ are the terms depending on the anomalous magnetic moments, the contributions in m units of the various parts to the mass difference when $q^2 = 0$ are distributed as follows:

	A_0	A_1	A_2
Neutron	-1.759	17.088	1.217
Proton	-0.853	11.704	0.010
ΔM	-0.906	5.384	1.207

It is interesting to notice that, were we to neglect completely the contributions coming from the core, we would obtain the values

	A_0	A_1	A_2
Neutron	-1.136	8.024	1.861
Proton	-1.374	3.095	1.192
ΔM	0.238	4.929	0.668

with a total ΔM of $5.836m$. This result would show that the core contributes negatively to the mass difference, contrary to our previous beliefs.

Finally we should remark that no attempt has been made in this calculation to take into account the effect of the experimental errors on the form factors; it may very well be possible that an adjustment of the constants given by HERMAN and HOFSTADTER as well as a better knowledge of the form factors themselves will improve our result. In particular a better knowledge of the core distributions represented at present by δ functions, would free the calculations from the unpleasant presence of divergences.

* * *

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Gamma Rays from ^{150}Eu (*).

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(ricevuto il 17 Giugno 1961)

The aim of the present work is to investigate the decay of $^{150}_{63}$Eu, produced by means of (γ, n) reactions. It is expected that bombarding europium with γ-rays, at least two neutron deficient isotopes, $^{150}_{63}$Eu and $^{152}_{63}$Eu, will be produced, the first one, originating from the $^{151}_{63}$Eu (γ, n) reaction (isotopic abundance 48%) and the second one from the $^{153}_{63}$Eu (γ, n) reaction (isotopic abundance 52%).

The decay of $^{152}_{63}$Eu is very well known (1) and occurs in both the adiacent isobars $^{152}_{62}$Sm (electron capture and positon decay) and $^{152}_{64}$Gd (negaton emission). This decay is a typical example of production of even-even nuclei lying in the boundary region between vibrational and rotational nuclei: ^{152}Sm ($N=90$) shows a well defined rotational structure, while ^{152}Gd ($N=88$) has a vibrational behaviour.

$^{150}_{63}$Eu is also expected to decay into both $^{150}_{62}$Sm and $^{150}_{64}$Gd, both from the β-decay energy systematics ($Q^+ \simeq 3$, $Q^- = 1.1$ MeV) (2) and from nuclidic mass computation ($Q^+ = 2.42$, $Q^- = 1.07$ MeV) (3).

A 15 h activity in the neutron deficient europium isotopes was reported for the first time by F. D. S. Butement (4), who bombarded europium with 23 MeV γ-rays. A more definite assignment was obtained by Wilkinson and Hicks (5) who bombarded samarium oxide with 10 MeV protons. The $(15.0 \pm \pm 0.5)$ h activity produced by (p, n) reactions was reported to decay with positon emission ($E_{max} = 18$ MeV) and with associated γ radiations up to 0.5 MeV and assigned to ^{150}Eu. Mack, Neuer and Pool (6) who produced ^{150}Eu by (p, n) reactions without chemical separations, have reported a 13.7 h activity decaying by negaton emission of 1.07 MeV maximum energy. No positon emission nor γ radiation have been observed in this case.

No excited levels are known in $^{150}_{64}$Gd

(*) Work performed under a contract stipulated between EURATOM and C.N.E.N.

(1) Nuclear Data Sheets NRC 59-4-82, 83, 84.

(2) Nuclear Data Sheets, NRC 59-2-55.

(3) F. Everling, L. A. Koenig, J. H. E. Mattauch and H. A. Wapstra: *Nucl. Phys.*, **18**, 529 (1960).

(4) F. D. S. Butement: *Nature*, **165**, 149 (1950).

(5) G. Wilkinson and H. G. Hicks: *Phys. Rev.*, **80**, 491 (1950).

(6) R. C. Mack, J. J. Neuer and M. L. Pool: *Phys. Rev.*, **91**, 903 (1953).

and the β^--decay of $^{150}_{63}$Eu is assumed to occur mostly to the $^{150}_{64}$Gd ground-state ([2]). For $^{150}_{62}$Sm many levels are known from the radioactive decay of $^{150}_{61}$Pm and from nuclear reaction data ([2]). It is shown below that also ^{150}Eu decays with associated γ-rays, most of them belonging to ^{150}Sm.

$^{150}_{62}$Sm ($N=88$) seems to show a not very well defined behaviour, from the point of view of the level structure. The low-lying levels at 340 and 780 keV are reported to have 2^+ and probably 4^+ spin, respectively, from angular correlation measurements ([7]), while the energy ratio $E(4^+)/E(2^+)=2.29$ is in quite good agreement with the empirical value (2 to 2.3) of the ratio $E(2'^+)/E(2^+)$ for vibrational nuclei.

Additional useful information on this situation may be obtained by studying in more details the decay of ^{150}Eu and the associated γ-ray spectrum. For this purpose the decay of ^{150}Eu was investigated, by producing this isotope with 31 MeV γ-rays from the betatron of the « Centro Studi Fisico-Biologici » in Turin on europium oxide.

The measured γ-activities showed essentially two half-lives of (9 ± 1) h and (14 ± 1) h, respectively. The first one is due to the known decay of 9.3 h $^{152}_{63}$Eu, obtained by $^{153}_{63}$Eu (γ, n) reaction, the second one to $^{150}_{63}$Eu obtained by the $^{151}_{63}$Eu (γ, n) reaction. Taking into account the production of ^{152}Eu, which has been confirmed by the presence of the known associated γ-rays, the assignment to mass 150 of the (14 ± 1) h activity is certain.

The γ-ray spectra were measured using scintillation techniques and displayed on a 200 channel LABEN analyzer The analysis of the spectra enabled us to report many γ-rays which can be assigned to the decay of ^{150}Eu in addition to the radiations due to ^{152}Eu. The results are reported in Table I, which shows the analysed γ-rays with their relative intensities and half-lives. The assignment to the mass 150 and 152 are also reported, and the already ([5]) known radiations from ^{152}Eu (9.3 h isomer) are also shown for comparison.

The relative intensities were normalized to the intensity of the strongest γ-ray, *i.e.* the 845 keV photopeak. However, it can be shown that this peak is not entirely due to the ^{152}Eu activity; first of all the half-life is longer than that of the 970 keV peak ((9.3 ± 0.3) h) which agrees with the known data for the decay of ^{152}Eu; secondly the $(845+$ $+122)$ keV cascade relationship, as pointed out below, indicates that a small fraction of the 845 keV peak is not due to the ^{152}Eu decay.

For these reasons the 845 keV peak is reported to consist of two different γ-rays with relative intensities 100 and 3 ± 1, respectively. The second one is assigned to the ^{150}Eu decay, in agreement with half-life considerations.

The same argument applies to the 345 keV peak for which the separation in two different γ-rays (one belonging to the ^{152}Eu, the other to the ^{150}Eu decay) has been made assuming the (12 ± 1) h half-life as an averaged value. These assignments are in agreement with the known level scheme of ^{150}Sm and with the already known relative intensities of the ^{152}Eu γ-rays.

We were not able to make a clear assignment for the two weak γ-rays at 550 and 1100 keV. However a 563 keV γ-ray is present in the decay of ^{152}Eu ([5]) ($\sim 2\%$), so it cannot be excluded that this could correspond to the 550 keV peak found in our measurements.

A qualitative investigation on possible γ-γ cascades was performed by measuring the summing scintillation spectrum with a well-type scintillation detector.

The $(842+122)$ keV cascade known in the decay of ^{152}Eu was clearly observed by the increasing of the $(970\pm$ $\pm10)$ keV peak in the summing spectrum

([7]) L. Rosler and C. A. Fenstermacher: *Bull. Am. Phys. Soc.*, 2, no. 5, 268L5; oral report.

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TABLE I.

Present investigation				^{152}Eu decay (9.3 h) ([5])	
E_γ (keV)	$T_{\frac{1}{2}}$ (h)	Assignment	Rel. int.	E_γ (keV)	Rel. intens.
40 ± 2 (X-ray)	9 ± 2	^{152}Eu+^{150}Eu	~ 250	—	~ 180 (calc. value)
123 ± 3	9 ± 1	^{152}Eu	~ 70	122 ± 1 (^{152}Sm)	69; 106
345 ± 5	12 ± 1	^{152}Eu ($\sim 53\%$) ^{150}Eu ($\sim 47\%$)	53 ± 3 ~ 28 ~ 25	344 ± 1 (^{152}Gd)	26 (adopt. value)
417 ± 5	13 ± 1	^{150}Eu	21 ± 3	410? (^{152}Sm)	weak
513 ± 5 ($\gamma^\pm$)	15	^{150}Eu	3.3 ± 0.6	—	—
550 ± 20	?	?	~ 1	563 ± 1 (^{152}Sm)	~ 2
845 ± 10	10.4 ± 0.4	^{152}Eu	100	842 ± 1 (^{152}Sm)	100
		^{150}Eu	3 ± 1	—	—
970 ± 10	9.3 ± 0.3	^{152}Eu	88 ± 6	963 ± 1 (^{152}Sm)	80 (adopt. value)
				970 ± 1 (^{152}Gd)	5 (adopt. value)
~ 1100	?	?	weak	—	—
1160 ± 30	~ 13	^{150}Eu	2.0 ± 0.5	—	—
1220 ± 30	~ 13	^{150}Eu	2.8 ± 0.6	—	—
1320 ± 30	9 ± 2	^{152}Eu	10 ± 1	1315 ± 1 (^{152}Gd)	10 (adopt. value)
1410 ± 30	9 ± 2	^{152}Eu	10 ± 1	1389 ± 1 (^{152}Sm)	10 (adopt. value)
1630 ± 60 (doub.)	16 ± 3	^{150}Eu	1.6 ± 0.7	—	—

and the corresponding lowering of the (845 ± 10) keV peak. However, the relative intensity of this latter was not in good agreement with the expected contribution to the summing spectrum.

In order to get a reasonable agreement, it must be assumed that a fraction of the 845 keV peak ($\sim 3\%$) is due to another type of decay. Because the half-life is not too different from 9 h, this fraction can be assigned to the 14 h ^{150}Eu decay. Two other summing peaks were observed at 760 keV and $\sim$1 270 keV, respectively. Because they cannot originate from γ-γ cascades in the ^{152}Eu decay, they were assigned to the ^{150}Eu spectrum. The first one originates from a (345+417) keV cascade, the second is probably due to the coincidence between the above cascade and the annihilation radiation of 511 keV. If these conclusions are correct the decay of ^{150}Eu will occurr by positon emission to a $(760 \pm \pm 7)$ keV level which in turn decays via the (417+345) keV γ-γ cascade.

If the value of 1.8 MeV for the positon energy as reported by WILKINSON and HICKS ([2]) corresponds to this branch, a disintegration energy of $\sim$ 3.6 MeV for the ^{150}Eu-^{150}Sm isobaric pair should be assumed.

Other levels at 1160, 1220, 1630 keV are expected in ^{150}Sm according to our preliminary results. They can be compared with the already known ([4,5]) levels at 1190 and 1670 keV, while the 845 keV γ-ray may be the 820 keV transition from a 2010 keV level to the 1190 keV level.

Further detailed investigations about the decay of ^{150}Eu are in progress in collaboration with the Instituut voor Kernphysisch Onderzoek in Amsterdam.

* * *

We want to express our heartiest thanks to Prof. G. WATAGHIN and Prof. B. BELLION for kindly placing at our disposal the betatron of the « Centro Studi Fisico-Biologici » in Turin, and to Prof. G. CORTINI for his constant interest and useful advice during the course of this work.

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Regularities in the Branching Ratios from the Higher Excited States of the Even-Even Nuclei (*).

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In the course of a study on the level schemes of a number of even-even nuclei — more specifically on the branching ratio from the second excited 2^+ state(1) — an interesting regularity was observed in the decay mode of the higher excited states. The more salient aspects are discussed below for a number of illustrative cases. The analysis of these cases follows the same general lines as developed in the basic and informative study by Scharff-Goldhaber and Weneser (2).

For most of the even-even nuclides, possessing a well developed vibrational structure — as demonstrated by the spacing, spin and branching ratio of the two lower 2^+ levels — it is observed that the γ-ray transition probability from a higher level to the first excited 2^+ state is in general greatly suppressed with respect to that to the second excited 2^+ state.

This result is most significant in those cases where the two γ-rays involved are pure multipoles as *e.g.* in the decay of a 4^+ level. Some of the higher excited states in ^{104}Pd, in ^{192}Pt and in ^{194}Pt form cases in point. Table I lists the relative reduced γ-ray transition probabilities from a number of levels decaying by (almost) pure $E2$ transitions to the two vibrational 2^+ states.

In those cases where the multipolarities of the γ-transitions are not known with certainty, the possibility exists that the two γ-rays of interest possess completely different multipole mixture ratios. It is somewhat surprising, however, as shown in Table II, that the reduced branching ratios deviate in the

(*) [A short report on this subject was presented by one of the authors (R.A.R.) at the XLV National Congress of the S.I.F. at Pavia, 1-7 Oct. 1959: *Suppl. Nuovo Cimento*, **1**, 141 (1960)].

(1) R. K. Girgis: *Thesis* (Amsterdam, 1959).

(2) G. Scharff-Goldhaber and S. Weneser: *Phys. Rev.*, **98**, 212 (1955).

TABLE I. – *Ratios of the reduced transition probabilities for (mainly) pure E2 γ-rays to the 2+ levels in some even-even nuclei.*

Nuclide 2^+ levels (keV)	Level		$\dfrac{B(E2; J\to 2^{+\prime})}{B(E2; J\to 2^+)}$
	E (keV)	J^π	
^{104}Pd	2090	4^+	80
2^+: 555	2200	4^+	50
$2^{+\prime}$: 1340	2270	4^+	> 300
^{192}Pt	921	3^+	80
2^+: 317	1201	3^+, 4^+	80
$2^{+\prime}$: 613			
^{194}Pt	923	3^+	> 20
2^+: 328	1267	0^+	11
$2^{+\prime}$: 622			
^{60}Ni	2505	4^+	~ 0.1
2^+: 1332			
$2^{+\prime}$: 2160			
^{116}Sn	2390	4^+	< 20
2^+: 1270	2535	4^+	> 4000
$2^{+\prime}$: 2090	2807	4^+	< 7.5
	3054	4^+	< 11

TABLE II. – *Ratio of the reduced transition probabilities from some higher energy levels to the 2+ levels in certain even-even nuclei.*

Nuclide 2^+ levels (keV)	Level		$\dfrac{B(E2; J\to 2^{+\prime})}{B(E2; J\to 2^+)}$	$\dfrac{B(M1; J\to 2^{+\prime})}{B(M1; J\to 2^+)}$
	E (keV)	J^π		
^{74}Ge	2200	2^+	170	70
2^+: 596				
$2^{+\prime}$: 1200				
^{76}Se	1770	2	175	37
2^+: 560	2080	—	54	16
$2^{+\prime}$: 1210	2400	2	60	25
	2640	3^+	11	5
^{110}Cd	2160	3^-	22	6
2^+: 656	2218	3^-, 4^+	24	6
$2^{+\prime}$: 1474				
^{112}Cd	2260	0, 1^+	5	2
2^+: 695	2430	—	21	8
$2^{+\prime}$: 1310	2820	—	12	5

same direction as those listed in Table I.

More specifically one concludes that the transition to the upper 2^+ member is enhanced and/or the one to the lower 2^+ member is suppressed. This is the same general behaviour as observed in the decay of the second excited 2^+ state in the vibrational nuclides to the first 2^+ state, respectively to the ground state, and generally explained as the effect of selection rules associated with the change in the vibrational quantum number n (0 for the ground state, 1 resp. 2 for the first resp. second 2^+ state). One might therefore be tempted to interpret the decay characteristics of the higher levels along these same lines. This cannot be the complete picture, however, as shown by the example of ^{104}Pd. Only one of the 4^+ levels can be associated with the $n=3$ band and should therefore show the observed characteristics; the others should behave quite differently.

One solution to this difficulty would be to assume configuration mixture between this so-called $n=3$ vibrational 4^+ state and a number of other neighbouring 4^+ levels which in zero approximation would have been described by completely different configurations.

This large amount of admixture seems to be completely different from the situation pertaining to the two lower 2^+ levels. It may indicate that at the region of excitation energies of the various 4^+ levels one has approached the limit of the region in which a collective vibrational description for most of the levels is still useful.

It has been remarked before in another connection (3), that the higher excited states of even-even nuclei seem to show a reluctance to decay into the energy gap. The overall shape of thermal neutron capture γ-ray spectra has been partially explained by assuming that levels slightly above (0.5 to 1.0 MeV) the upper limit of the energy gap, decay preferentially to the top of this gap and less frequently to levels within the gap region. In that specific instance, rotational nuclei were involved and the explanation forwarded assumed that K-forbiddeness played a major rôle. This same explanation can hardly hold for the nuclei listed in Table I and II and the observed regularity therefore seems to be of a more general nature.

Independently, BÄCKSTRÖM *et al.* (4) have noticed a behaviour, similar to that described above, in the decay of the higher levels of ^{194}Pt and the levels discussed by these authors are also listed in Table I.

Nuclides with a closed shell are expected to show a far more individual behaviour. Unfortunately, due to the effect of shell closure, the second 2^+ level of interest lies at a rather high excitation energy and the γ-ray branch from higher levels to this second 2^+ level is therefore very weak in most cases anyway. A clear illustrative case is presented by the decay of the 2.50 MeV 4^+ level of ^{60}Ni (see Table I).

A closed shell nuclide which clearly demonstrates the difficulties in interpretation is ^{116}Sn. As in ^{104}Pd, there exists a number of 4^+ states, populated in the decay of 54 min ^{116}In. These levels decay to two 2^+ states and the reduced branching ratios are also shown in Table I. From this evidence it might be concluded that even in this case the same general trend can exist although a pure single particle behaviour cannot be excluded. There is one level, however, where branching to the second 2^+ level is extremely favoured.

There are collective vibrational aspects in the level structure of ^{116}Sn as well as more individual particle ones.

(3) L. V. GROSHEV, A. M. DEMIDOV and V. I. PELEKHOV: *Nucl. Phys.*, **16**, 645 (1960). V. M. STRUTINSKY, L. V. GROSHEV and M. K. AKIMOVA: *Nucl. Phys.*, **16**, 657 (1960).

(4) G. BÄCKSTRÖM, O. BERGMAN, I. BURDE and I. LINDSKOG: *Nucl. Phys.*, **15**, 566 (1960).

Coulomb excitation has shown the first 2^+ state at 1.270 MeV and a 3^- level at 2.23 MeV to be of a collective character. The second vibrational 2^+ level has not shown up so far in these experiments.

In fact, there are two candidates for the second vibrational 2^+ level, one at 2090 and one at 2180 keV, both with spins 2^+, the lower one has been used in Table I and is populated in the decay of 54 min ^{116}In, the higher one is only fed from 15 min ^{116}Sb; in both cases the other member is conspicuously absent. This could be due to the fact that these two 2^+ levels have a completely different configuration. If this is indeed the case two conclusions can be drawn: firstly, it is not clear which of the two is the more appropriate one to be identified as the second vibrational 2^+ level, and secondly, even in these two close-lying 2^+ states the amount of configuration mixing must be small.

The arguments developed here seem to show an attractive interest in the establishment of definite systematic properties in the deexcitation of the levels above the low well established vibrational structure in even-even nuclei.

Further detailed analyses about this subject are in progress.

On Radiative Corrections due to Soft Photons.

(*Nuovo Cimento*, **19**, 1010 (1961))

K. E. Eriksson
CERN - *Geneva*

	Errata	Corrige
p. 1015, eq. (14)	$\prod_{i=1}^{n}(\dots)\sum_{(\dots)}\cdots$	$\prod_{j=1}^{n}(\dots)\sum_{(\dots)}\cdots$
p. 1015, eq. (14)	$\prod_{j=1}^{n}\left(\frac{ieQ_j p_i^{\mu_j}}{k_j\cdot p_i}\right)\hat{M}$	$\prod_{j=1}^{n}\left(\frac{ieQ_i p_i^{\mu_j}}{k_j\cdot p_i}\right)\hat{M}$
p. 1015, eq. (16)	$\varepsilon_i=\begin{cases}+1\\-1\end{cases}$	$\varepsilon_i=\begin{cases}-1\\+1\end{cases}$
p. 1024, footnote	$\Delta E/P_{i0}$	$\Delta E/p_{i0}$

Radiative Corrections to Muon-Electron Scattering.

(*Nuovo Cimento*, **19**, 1029 (1961))

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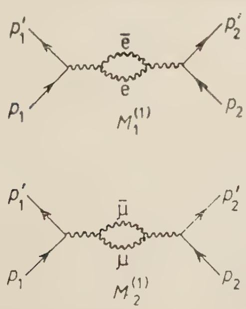
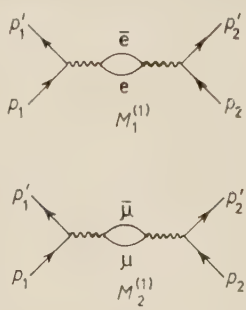
	Errata	Corrige
p. 1032, Fig. 3		

	Errata	Corrige
p. 1035, eq. (30)	$R_3 + R_4 = \frac{2m_1^2 m_2^2}{\Delta^6} \mathrm{Re}\{F_{1\mu}F_2^{\mu}\} \cdot \sum_{i=1}^{2} \frac{1}{\sqrt{1+t_i}} \ln \frac{\sqrt{1+t_i}+1}{\sqrt{1+t_i}-1}$.	$R_3 = \frac{m_1^2}{\Delta^6} \frac{1}{\sqrt{1+t_1}} \ln \frac{\sqrt{1+t_1}+1}{\sqrt{1+t_1}-1} \cdot (p'_{1\mu} + p_{1\mu}) \mathrm{Re}\{iF_{1\nu}\} F_2^{\mu\nu}$ (similarly for R_4)
p. 1035, eq. (33)	$R_3 + R_4 = \frac{1}{8} t_1 t_2 (2u-1) \cdot \sum_{i=1}^{2} \frac{1}{\sqrt{1+t_i}} \ln \frac{\sqrt{1+t_i}+1}{\sqrt{1+t_i}-1}$.	$R_3 + R_4 = \frac{1}{2} \sum_{i=1}^{2} [u(u-1) - \frac{1}{4} t_i] \cdot \frac{t_i}{\sqrt{1+t_i}} \ln \frac{\sqrt{1+t_i}+1}{\sqrt{1+t_i}-1}$
p. 1039, eq. (52)	$a = \frac{1}{\sqrt{\cdots}} \cdot \sum_{i=1}^{2} \left[\ln \frac{(\ldots)^2}{(u+T)} - \ln \frac{t_i}{4}\right]$	$a = -\frac{1}{\sqrt{\cdots}} \cdot \sum_{i=1}^{2} \left[\ln \frac{(\ldots)^2}{4(u+T)} - \ln \frac{t_i}{4}\right]$
p. 1040, Sect. 4 second line	involves the small factor $\frac{1}{8} t_1 t_2$	is always unimportant and tends to zero like m_i^2/Δ^2 for large Δ
p. 1040, eq. (58)	$+ \ln \left\| \frac{u+T}{1-u+T} \right\| - 2 \ln \frac{u+\frac{1}{4}t_i}{1-u+\frac{1}{4}t_i}$	$+ Q \ln \frac{u-1}{u}$
p. 1041, eq. (61)	$\left\| \Phi\left(\frac{E_i - q}{2q}\right) - \Phi\left(\frac{E_i + q}{2q}\right) \right\|$	$\left\| \Phi\left(-\frac{E_i - q}{2q}\right) - \Phi\left(\frac{E_i + q}{2q}\right) - \frac{1}{2}\pi^2 \right\|$
p. 1042, eq. (64)	$\delta = \ldots H_1(u)$.	$\delta = \ldots H_1(u),\ Q = 0$.
p. 1042, footnote	eq. (35), (24), (25)	eq. (35.24), (35.25)

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LIBRI RICEVUTI E RECENSIONI

P. PENFIELD jr. – *Frequency-Power Formulas.* Ed. John Wiley and Sons, New York e Londra; pagine VIII-168, $ 4.00.

Questo libro contiene la prima analisi sistematica delle formule « frequenza-potenza ». Esse stabiliscono delle relazioni fra le potenze entranti in un sistema a diverse frequenze e le frequenze stesse; la loro importanza sta principalmente nel fatto che esse continuano ad esser valide anche quando il sistema in esame contiene elementi non lineari, quando cioè molti dei metodi normali di indagine non sono più applicabili.

Vi sono quattro tipi fondamentali di formule « frequenza-potenza » e riguardano rispettivamente:

1) le potenze attive in reattanze non lineari;

2) le potenze reattive in reattanze non lineari;

3) le potenze attive in resistenze non lineari;

4) le potenze reattive in resistenze non lineari.

Le più importanti, per le varie applicazioni a cui si prestano, sono quelle del 1° tipo, introdotte da Manley e Rowe per la prima volta, e successivamente applicate da loro e da altri all'analisi di tutta una serie di sistemi che vanno dalle macchine elettriche agli amplificatori magnetici, agli amplificatori parametrici, ai maser ed ai plasmi.

L'autore riporta prima i vari tipi di formule nella loro espressione originaria con le relative dimostrazioni, poi prende in esame formule più generali per i sistemi a costanti concentrate e le dimostra; ne individua quindi i limiti di validità, portando vari esempi interessanti di sistemi che obbediscono ad un tipo di formula od all'altro. Passa poi all'analisi dei sistemi continui per i quali sia ancora possibile definire le potenze entranti nel sistema in varie zone del contorno a diverse frequenze ed arriva alla conclusione che le formule valgono per tutti i sistemi fisici per i quali si può introdurre un'energia come funzione di stato e per i sistemi continui che obbediscono al principio di Hamilton. Porta quindi vari esempi di notevole interesse (fascio di elettroni in moto irrotazionale - mezzo giromagnetico - Flusso rotazionale di un liquido - fluido magnetoidrodinamico) e fa vedere come si possono ricavare per questi sistemi le formule di Manley e Rowe direttamente dalle equazioni generali di moto e di propagazione nei casi più semplici, oppure partendo sistematicamente dalle equazioni di Hamilton nei casi più complessi.

Seguono infine alcuni esempi di applicazione delle formule alle macchine elettriche ed alle reti per telecomunicazioni.

Le formule « potenza-frequenza » sono molto utili nell'analisi dei sistemi in cui avvengono conversioni di frequenza e servono sia per arrivare al risultato nel singolo caso particolare, sia come strumento per l'indagine e l'intuizione del comportamento fisico del sistema in esame. L'autore sottolinea che esse sono in sostanza dei nuovi principi di conser-

vazione ed è quando sono usate come tali che esse rivelano la loro efficacia.

Il libro è un contributo veramente notevole alla teoria ed alla interpretazione di questo nuovo tipo di formule che costituiscono un mezzo molto utile per affrontare nuovi problemi e per guardare da un nuovo punto di vista i problemi già risolti con metodi convenzionali.

A. SONA

E. M. ROGERS – *Physics for the Inquiring Mind*; Princeton University Press, Princeton, N. J., 1960; pp. X-778.

Un sottotitolo di questo libro, stampato nell'interno, porta la scritta: *Metodi, natura e filosofia della scienza fisica*, e veramente il libro tenta di presentare insieme tutto questo. Esso è un testo di fisica scritto per l'insegnamento di un corso annuale di fisica (precisamente il corso che già da molti anni si tiene a Princeton) a una classe di studenti di diverse provenienze e di diverse facoltà — medicina, materie letterarie, economia, scienze biologiche — e per questo suo speciale carattere non presuppone nè una preparazione precedente specializzata, nemmeno a livello liceale, nè una particolare conoscenza della matematica: di essa si serve solo come di uno strumento, e le conoscenze richieste non vanno oltre l'algebra elementare e la geometria piana, come sono insegnate anche da noi nei licei classici; d'altra parte invece, per la lettura del libro, e molto più per la risoluzione dei numerosissimi problemi proposti, si richiedono capacità di lettura critica, buone facoltà di giudizio e di ragionamento, e chiarezza di idee.

Il libro non tratta « tutta » la fisica estesamente, ma solo approfondisce un certo numero di argomenti, scelti in modo da dare una idea ragionevole della struttura logica della fisica come si presenta attualmente. Per questa ragione sono omessi, o trattati solo superficialmente alcuni argomenti tradizionalmente molto più sviluppati nei testi analoghi: per esempio la statica, la idrostatica, la calorimetria, l'ottica geometrica, l'acustica, e alcune parti della elettrologia e della magnetostatica; sono invece trattate a fondo la dinamica, l'astronomia planetaria, la teoria molecolare, parte dell'elettricità e del magnetismo, e la « fisica atomica », continuamente intrecciata alle discussioni di carattere generale.

Più particolarmente, il libro è suddiviso in cinque parti e due « interludi ». I titoli delle cinque parti sono: 1) materia, moto e forza; 2) astronomia: storia di una teoria; 3) molecole ed energia; 4) elettricità e magnetismo; 5) fisica nucleare ed atomica. I due « interludi » riguardano un'appendice sull'aritmetica (che introduce i calcoli mediante le potenze di dieci, il calcolo degli errori, delle percentuali, delle stime, delle approssimazioni, le rappresentazioni grafiche delle relazioni di proporzionalità diretta e inversa, le medie ponderali, e le interpolazioni) e una seconda appendice che tratta da un punto di vista generale della evoluzione dei metodi e dei concetti della matematica, della relatività, e della loro applicazione nella descrizione e nella interpretazione della fisica nucleare: questo interludio porta come titolo « matematica e relatività ».

Dei vari paragrafi di cui ogni capitolo si compone, alcuni sono ritenuti essenziali, altri da trattarsi in modo meno approfondito, altri addirittura da omettere nel caso che non si abbia un particolare interesse agli argomenti in essi trattati (questi ultimi riguardano: la tensione superficiale e l'idrodinamica; l'astronomia del diciassettesimo secolo; la chimica e l'elettrolisi).

Gli argomenti delle esperienze da eseguire nel lavoro di laboratorio sono intercalati al testo e ai problemi di cui alcune volte sono un complemento o

una preparazione; lo scopo di tale lavoro è dichiaratamente ristretto all'acquisto della conoscenza di come realmente lavorano gli scienziati. Se a questa conoscenza avverrà che naturalmente si aggiunga un vivo interesse e un senso di gioiosa conquista, allora, e solo allora, il lavoro di laboratorio e le abilità in esso acquistate potranno dare un duraturo beneficio allo sviluppo della personalità. Le esperienze sono soltanto suggerite: non vi è traccia del metodo spesso chiamato ironicamente « cook-book », che dà le prescrizioni degli oggetti da usare, delle operazioni da eseguire, e i risultati che si debbono trovare.

I problemi hanno una parte importante nella didattica del libro, e sono studiati non tanto in modo da richiedere risultati numerici, quanto dei ragionamenti e delle discussioni; sono anche inserite nel testo, come copie di fogli dattiloscritti, delle tracce per la risoluzione di alcuni problemi, sotto forma di domande, le risposte alle quali, se corrette, conducono per gradi alla soluzione dell'intero problema: questo metodo appare didatticamente molto efficace.

Il testo è accuratissimo, e strettamente rigoroso nelle sue deduzioni e nella concatenazione logica dei vari argomenti. Non è facile trovarvi inesattezze, ma forse la spiegazione di qualche punto è appesantita oltre il necessario dal moltiplicarsi degli argomenti a favore o contro una data teoria. Questa osservazione non vuole però naturalmente riferirsi a quella che invece può considerarsi una delle migliori qualità del libro, di non limitare gli esempi a un solo ristretto campo, ma di prenderli da tutta l'intera costruzione della fisica, e facendo spesso ricorso alla esperienza (anche vagamente tecnica) della vita di ogni giorno.

La forma generale della espressione è chiara ed arguta, estremamente semplice, ma impreziosita da citazioni letterarie o filosofiche, che fanno volutamente contrasto con l'apparente ingenuità di alcuni schizzi e figurine, animate da ometti schematici, e perfino da diavoletti con corna e coda.

Questo bel libro, che speriamo di vedere presto tradotto in italiano, sarà utilissimo, oltre che ad ogni persona genericamente colta che non voglia troppo sfigurare in un mondo dalla mentalità scientifica, agli studenti di fisica, per aiutarli a chiarirsi le idee fondamentali e a sviluppare un sano senso critico su vari problemi. Ma la massima utilità dalla lettura, anzi dallo studio e dall'uso quotidiano di questo libro, potranno ritrarla gli insegnanti di fisica di scuole secondarie, che vi troveranno una inesauribile fonte di argomenti adatti a chiarire i concetti e i metodi della fisica alle giovani menti non ancora familiari con questa disciplina, e viziate spesso da una « forma mentis » assolutamente antiscientifica, difficilissima da vincere, ma la cui dannosa influenza è una delle maggiori cause di molti insuccessi universitari.

M. FERRETTI

R. J. STEPHENSON – *Mechanics and properties of matter*, 2nd ed.; John Wiley and Sons, New York, 1960; pp. x-367; $ 7.50.

Il libro è dedicato agli studenti di fisica ed ingegneria dei primi anni, dopo un anno di fisica generale ed un anno di analisi matematica. Esso corrisponde secondo l'autore, ad un corso semestrale con tre ore settimanali di lezioni e due o tre ore di esercitazioni. Un corso serrato, dunque, dove non si può perder tempo.

L'ordine della materia è piuttosto classico, a parte alcune novità che noteremo: Cinematica, Forze, Moti, Lavoro, Energia di una particella: Legge di gravitazione di Newton; Oscillazioni armoniche; Moti rigidi; Proprietà dei solidi e dei liquidi; Statica; Moti ondosi.

Ogni capitolo è corredato da esercizi

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che non mancano di originalità ed hanno in generale una simpatica presentazione.

I primi otto esercizi del primo capitolo sono di cinematica relativistica e le trasformazioni di Lorentz (di Einsten, nel libro) sono date già a p. 3, in una nota. Attenti come siamo, in quest'epoca, alla questione dell'insegnamento della meccanica relativistica (quanto, come, quando), abbiamo seguito l'autore su questo punto con una certa attenzione, ma ne siamo rimasti un poco delusi: la nota a piè di pagina è solo seguita da un pur chiaro ma isolato paragrafo a p. 78, e poco altro: uno spiraglio che si apre e che si chiude troppo presto per essere un'apertura, e che sa di aggiunta e rimedio. Ed è un peccato, perchè l'autore sembra avere le qualità per invitare e divulgare, ed ha chiaramente una sensibilità moderna, agile ma non trascurata. Sono suoi pregi un leggero ma costante inquadramento storico; un periodare breve e con definizioni precise; eleganza e chiarezza di disegni, di notazioni, di carattere tipografico, secondo le tradizioni della letteratura universitaria americana (o nostre latine dispense!).

Ma da un punto e non minore vi è sul quale dobbiamo avvertire almeno il lettore italiano. Nel titolo si promette, oltre la meccanica, un discorso sulle proprietà della materia, mentre questo sostanzialmente non c'è. È vero che si trattano le proprietà dei solidi e dei liquidi e che si dà la legge di gravitazione universale, ma non più e non meno di quanto non si faccia da noi nei corsi migliori di meccanica razionale. Non si parli allora di proprietà della materia, perchè nel 1960 questo titolo è di altro impegno, e presuppone tutta una impostazione che questo testo non ha: la teoria cinetica dei gas, la definizione almeno di ciò che per gas si intende, l'atomo, la molecola, ecc. Altrimenti aumentiamo con l'equivoco una scissione tradizionale e che ormai da troppo continua: la materia dei meccanici, newtoniana, e quella dei fisici, coulombiana, atomica.

Libro di meccanica, dunque, e nei suoi termini tradizionali, quello che qui si presenta: come tale è pregevole e lo si può con fiducia consigliare ai giovani studenti.

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